International Journal of Bifurcation and Chaos, Vol. 12, No. 8 (2002) 1779–1788

BISHADOWING AND HYPERBOLICITY

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Received March 28, 2001; Revised September 4, 2001

Shadowing of a dynamical system is often used to justify the validity of computer simulations of the system, and in numerical calculations an inverse form of the shadowing concept is also of some interest. In this paper we characterize the notion of shadowing in terms of stability, and express the notion of hyperbolicity using the concept of inverse shadowing.

Keywords: Bishadowing; hyperbolic; inverse shadowing; shadowing; stable; structurally stable.

1. Introduction

Shadowing, or the pseudo orbit tracing property (POTP), of a dynamical system is often used to justify the validity of computer simulations of the system, asserting that there is a true orbit of the system close to the computed one. The property was first established for systems generated by hyperbolic diffeomorphisms and later for those generated by hyperbolic homeomorphisms. The theory of shadowing has been intensively developed in recent years and has become a significant part of the qualitative theory of dynamical systems, with many interesting and deep results [Pilyugin, 1999].

In numerical calculations an inverse form of the shadowing concept is also of some interest: *Can every orbit of the system be shadowed by a numerical trajectory calculated by the specific computational routines and procedures under consideration?* A composite concept of bishadowing, combining both shadowing and inverse shadowing, has been introduced and shown to hold for systems generated by semi-hyperbolic mappings [Diamond *et al.*, 1993, 1995; Kloeden *et al.*, 1999]. An appropriate choice of the class of admissible pseudo orbit is crucial here. For example, Corless and Pilyugin [1995] showed that diffeomorphisms with the strong transversality condition do not have the inverse shadowing property if this class is too large, while it has been shown that Morse–Smale diffeomorphisms have the inverse shadowing property with respect to an appropriately restricted class of pseudo orbits, but do not have the property with respect to larger classes of pseudo orbits [Lee & Choi, 2002]. Very recently Hunt showed that the orbits of a dynamical system can be approximated by pseudo orbits which are interpreted as sample paths of a suitable Markov chain based on a finite partition of phase space [Hunt, 2001].

In Sec. 2, we introduce the concept of shadowing of a homeomorphism on a compact metric space X with respect to various classes of admissible

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Supported by the Korea Research Foundation Grant (KRF-2000-DA0015).

pseudo orbits. It is then shown that the notion of shadowing is independent of the choice of the class of admissible pseudo orbits whenever X is a compact manifold. Moreover we characterize the concept of shadowing in terms of stability.

In Sec. 3, we express the notion of hyperbolicity in terms of the concept of inverse shadowing. It is well known that an expansive homeomorphism with the shadowing property on a compact manifold is topologically stable [Walters, 1978]. Hence, it has the inverse shadowing property with respect to a suitable class of pseudo orbit [Kloeden & Ombach, 1997]. However the expansivity and the inverse shadowing property of a homeomorphism on a compact manifold does not imply the shadowing property. In fact, the pseudo Anosov diffeomorphism on a compact surface is expansive and has the inverse shadowing property with respect to a suitable class of pseudo orbit [Lewowicz, 1983, Corollary 3.1], but does not have the shadowing property.

We introduce the concept of stable closed expansivity which is slightly stronger than that of expansivity and show that a diffeomorphism on a compact manifold is Anosov if and only if it is stably closed expansive and has the inverse shadowing property with respect to a suitable class of pseudo orbits. Moreover we claim that a diffeomorphism is Anosov if and only if it has the weak inverse shadowing uniqueness property with respect to a suitable class of pseudo orbits.

2. Shadowing and Stability

Let (X, d) be a compact metric space and let $Z(X)$ denote the space of homeomorphisms on X. A homeomorphism will be identified with the dynamical system it generates by iteration. Define a metric d_0 on $Z(X)$ by

$$d_0(f, g) = \sup\{d(f(x), g(x)), d(f^{-1}(x), g^{-1}(x)) : x \in X\},$$

for any $f, g \in Z(X)$.

A *δ-pseudo orbit* of $f \in Z(X)$ is a sequence of points $\xi = \{x_k \in X : k \in \mathbb{Z}\}$ such that

$$d(f(x_k), x_{k+1}) < \delta, \quad k \in \mathbb{Z}.$$

A δ-pseudo orbit $\xi = \{x_k \in X : k \in \mathbb{Z}\}$ is said to be *ε-shadowed* by a point $x \in X$ (or an orbit $\{f^k(x) : k \in \mathbb{Z}\}$ if

$$d(f^k(x), x_k) < \varepsilon, \quad k \in \mathbb{Z}.$$

Say that $f \in Z(X)$ has the *pseudo orbit tracing property* (POTP) if given $\varepsilon > 0$ there exists $\delta > 0$ such that any δ-pseudo orbit of f is ε-shadowed by a point in X.

Let $X^{\mathbb{Z}}$ be the compact space of all two sided sequences $\xi = \{x_n : n \in \mathbb{Z}\}$ with elements $x_n \in X$, endowed with the product topology. For $\delta > 0$, let $\Phi_f(\delta)$ denote the set of all δ-pseudo orbits of f.

A mapping $\varphi : X \to \Phi_f(\delta) \subset X^{\mathbb{Z}}$ is said to be a *δ-method* for f. Then each $\varphi(x) \in \Phi_f(\delta)$ is a δ-pseudo orbit of f. For convenience, write $\varphi(x)$ for $\{\varphi(x)_k\}_{k\in\mathbb{Z}}$. Say that φ is a *continuous δ-method* for f if φ is continuous. The set of all δ-methods (resp. continuous δ-methods) for f will be denoted by $\mathcal{T}_0(f, \delta)$ (resp. $\mathcal{T}_c(f, \delta)$), and $\mathcal{T}_0(f)$ and $\mathcal{T}_c(f)$ are defined as

$$\mathcal{T}_0(f) = \bigcup_{\delta>0} \mathcal{T}_0(f, \delta)$$

and

$$\mathcal{T}_c(f) = \bigcup_{\delta>0} \mathcal{T}_c(f, \delta),$$

respectively. Moreover, every $g \in Z(X)$ with $d_0(f, g) < \delta$ induces a continuous δ-method $\varphi_g : X \to X^{\mathbb{Z}}$ for f by defining

$$\varphi_g(x) = \{g^k(x) : k \in \mathbb{Z}\}.$$

Let $\mathcal{T}_h(f)$ denote the set of all continuous methods for f which are induced by homeomorphisms on X. Clearly,

$$\mathcal{T}_h(f) \subset \mathcal{T}_c(f) \subset \mathcal{T}_0(f).$$

Note that a method in $\mathcal{T}_c(f)$ need not be generated by a single mapping.

Definition 2.1. $f \in Z(X)$ has the *shadowing property with respect to the class $\mathcal{T}_\alpha$, $\alpha = 0$, c, h,* if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any δ-method φ in $\mathcal{T}_\alpha(f)$ and any point $y \in X$ there exists a point $x \in X$ for which

$$d(f^k(x), \varphi(y)_k) < \varepsilon, \quad k \in \mathbb{Z} \text{ (see Fig. 1)}.$$

Definition 2.2. $f \in Z(X)$ has the *inverse shadowing property with respect to the class $\mathcal{T}_\alpha$, $\alpha = 0$, c, h,* if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any δ-method φ in $\mathcal{T}_\alpha(f)$ and any point $x \in X$ there exists a point $y \in X$ for which

$$d(f^k(x), \varphi(y)_k) < \varepsilon, \quad k \in \mathbb{Z} \text{ (see Fig. 1)}.$$

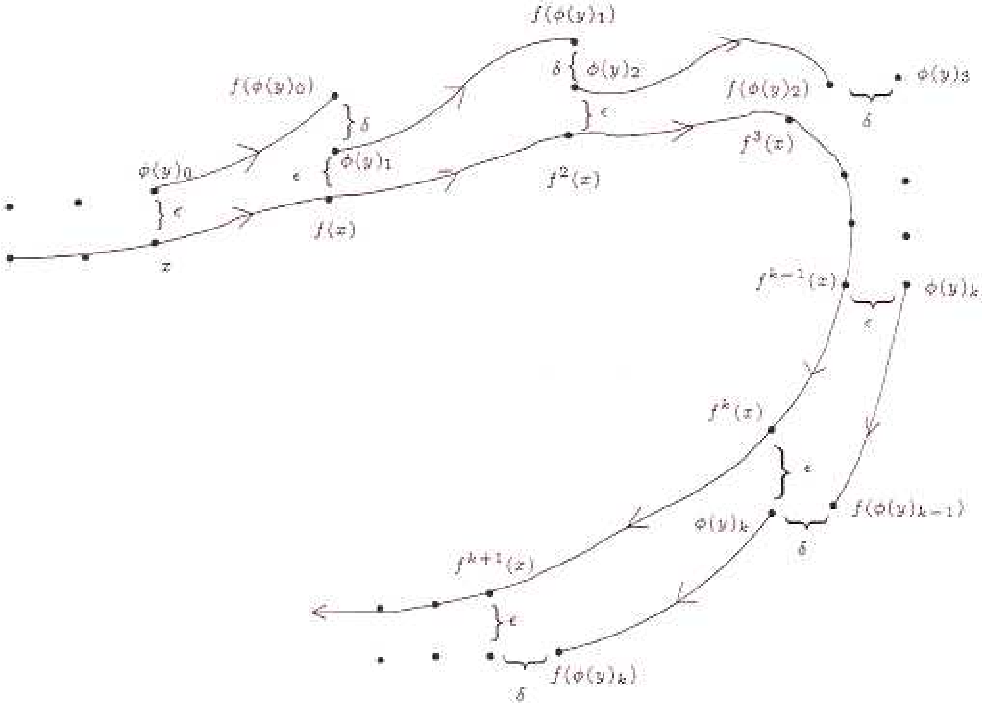

Fig. 1. An δ-pseudo orbit $\phi(y)$ is ϵ-shadowed by an orbit $\{f^k(x)\}$; and an orbit $\{f^k(x)\}$ is ϵ-shadowed by an δ-pseudo orbit $\phi(y)$.

Remark 2.3. Now, $f \in Z(X)$ has the shadowing property with respect to the class $\mathcal{T}_0$ if and only if f has the POTP. To see this, let $\xi = \{x_k \in X : k \in \mathbb{Z}\}$ be a δ-pseudo orbit of f. Then the map $\varphi : X \to X^{\mathbb{Z}}$ defined by

$$\varphi(x)_k = \begin{cases} x_k & \text{if} \quad x = x_0, \\ f^k(x) & \text{if} \quad x \neq x_0 \end{cases}$$

is a δ-method for f.

It is clear that if $\mathcal{T}_{\alpha_1} \subset \mathcal{T}_{\alpha_2}$ then the shadowing (resp. inverse shadowing) property with respect to the class $\mathcal{T}_{\alpha_1}$ is weaker than that of the shadowing (resp. inverse shadowing) property with respect to the class $\mathcal{T}_{\alpha_2}$, where $\alpha_1, \alpha_2 \in \{0, c, h\}$.

In all that follows, M denotes a compact smooth manifold and $\mathrm{Diff}(M)$ denotes the space of C^1 diffeomorphisms on M with the C^1 topology.

Corless and Pilyugin [1995] proved that every $f \in \mathrm{Diff}(M)$ with the strong transversality condition does not have the inverse shadowing property with respect to the class $\mathcal{T}_0$. Kloeden and Ombach [1997] have shown that every Anosov diffeomorphism has the inverse shadowing property with respect to the class $\mathcal{T}_c$. Very recently, Lee and Choi [2002] claimed that Morse–Smale diffeomorphisms have the inverse shadowing property with respect to an appropriately restricted class of pseudo orbits, but do not have the property with respect to larger classes of pseudo orbits.

The principal result of this paper is that the shadowing property for a homeomorphism $f \in Z(X)$ is independent of the choice of the class of admissible pseudo orbits if X is a compact manifold. The assumption that the phase space X is a manifold cannot be removed as the following example shows.

Example 2.4. Consider the circle $S = \{(\xi, \eta) : (\xi - 1/2)^2 + \eta^2 = 1/4\}$, coordinatized by $\theta \in [0, 1)$, and define the homeomorphism f_1 on S as follows:

$$f_1(\theta) = \theta \quad \text{if} \quad \theta = 0 \quad \text{or} \quad \theta = \frac{1}{2};$$

$$f_1(\theta) < \theta \quad \text{if} \quad \theta \in \left(0, \frac{1}{2}\right);$$

$$f_1(\theta) > \theta \quad \text{if} \quad \theta \in \left(\frac{1}{2}, 1\right).$$

Let $L = \{(x, 0) \in R^2 : 0 \leq x \leq 1\}$, and consider the homeomorphism f_2 on L given by $f_2(x, 0) = (x^2, 0)$. Define a homeomorphism f on $X = S \cup L$ by

$$f(x) = \begin{cases} f_1(x) & \text{if} \quad x \in S, \\ f_2(x) & \text{if} \quad x \in L. \end{cases}$$

It is easily checked that f has the shadowing property with respect to the class $\mathcal{T}_h$, but does not have the shadowing property with respect to the class $\mathcal{T}_0$.

Indeed, rather more can be shown.

Theorem 2.5. *A homeomorphism f on S has the shadowing property with respect to the class $\mathcal{T}_0$ if and only if it has the shadowing property with respect to the class $\mathcal{T}_h$.*

To prove this, some further notations and a known result (Lemma 2.6) are required. When studying the theory of shadowing of homeomorphisms on S, without loss of generality, attention can be restricted to those homeomorphisms on S which preserve orientation.

Let (X, d) be the circle S, with coordinate $x \in [0, 1)$, and with d the metric on S induced by the usual distance on the real line. It is easy to show that $f \in Z(S)$ has the shadowing (resp. inverse shadowing) property with respect to the class $\mathcal{T}_\alpha$, $\alpha = 0, c, h$, if and only if $f^k \in Z(S)$, with some $k \in \mathbb{Z}$, has the shadowing (resp. inverse shadowing) property with respect to the same class $\mathcal{T}_\alpha$.

Let $\pi : \mathbb{R} \to S$ be the covering projection defined by the relations

$$\pi(x) \in [0, 1), \quad \pi(x) \equiv x (\mathrm{mod}\ 1)$$

with respect to the coordinate x on S. Let $F : \mathbb{R} \to \mathbb{R}$ be the lift of f such that $F(0) \in [0, 1)$. It is well known that for any $x \in \mathbb{R}$ the limit

$$\mu(f) = \lim_{n -> \infty} = \frac{F^n(x)}{n} (\mathrm{mod}\ 1)$$

exists and does not depend on x. This quantity is called the *rotation number* of f and measures the average amount that a point in S is rotated by f. The main property of the rotation number is that f has a periodic point if and only if $\mu(f)$ is rational.

Plamenevskaya [1997] gave necessary and sufficient conditions under which a homeomorphism of S has the POTP.

Lemma 2.6 [Plamenevskaya, 1997]. *$f \in Z(S)$ has the POTP if*

1. *$\mathrm{Fix}(f)$ is nowhere dense and contains at least two points,*
2. *for any $a, b \in \mathrm{Fix}(f)$, either $\mathrm{Fix}(f) \cap (a, b) = \emptyset$ or the function $F(t) - t$ changes sign on (a, b),*

where $\mathrm{Fix}(f)$ is the set of all fixed points of f, F is the lift of f such that $F(0) \in [0, 1)$, and (a, b) is the open arc of S corresponding to the set $(a, b) \subset [0, 1)$ whenever $a < b$.

Theorem 2.5 will follow from this lemma. The proof uses techniques of [Plamenevskaya, 1997].

Proof of Theorem 2.5. Suppose $f \in Z(S)$ has the shadowing property with respect to the class $\mathcal{T}_h$. Then it is clear that $\mathrm{Fix}(f)$ is nowhere dense. First, f has periodic points. For, if not the rotation number $\mu(f)$ is irrational and we can choose $0 < \varepsilon < 1/4$ such that

(i) $d(x, f(x)) > 3\varepsilon$,

(ii) $d(x, y) < \varepsilon$ implies $d(f(x), f(y)) < \dfrac{1}{4}$,

for all $x, y \in S$. For each such ε, let $\delta = \delta(\varepsilon) > 0$ be that corresponding to ε in the shadowing property 2.1 for f with respect to the class $\mathcal{T}_h$. Let $x \in S$. Choose $0 < \delta' < \delta$ and $m, n \in \mathbb{Z}$, $m < n$, such that

$$d(f^m(x), f^n(x)) < \delta'$$

and

$$d(f^m(x), f^p(x)) \geq \delta'$$

for all $m < p < n$. Let $f^m(x) = x_0$, and put $N = n - m$. Choose an arc (a, b) in S such that

$$x_0, f^N(x_0) \in (a, b) \quad \text{and} \quad d(a, b) = \delta'.$$

Select a homeomorphism $h \in Z(S)$ such that

$$h(f^N(x_0)) = x_0; \quad \text{and} \quad h(x) = x \text{ if } x \notin (a, b).$$

Put $g = h \circ f$. Then

$$d_0(f, g) < \delta' \quad \text{and} \quad g^N(x_0) = x_0 .$$

By assumption, there exists a point $y \in S$ satisfying

$$d(f^k(y), g^k(x_0)) < \varepsilon$$

for all $k \in \mathbb{Z}$. Since $d(x_0, y) < \varepsilon$, there are $x_0', y' \in [0, 1)$ such that

$$|x_0' - y'| < \varepsilon, \pi(x_0') = x \quad \text{and} \quad \pi(y') = y .$$

Let F and G be the lifts of f and g, respectively, satisfying $F(y')$, $G(x_0') \in [0, 1)$. Then it is easily seen that

$$|F^k(y') - G^k(x_0')| < \varepsilon$$

for all $k \in \mathbb{N}$. For, if this is not so there exists a smallest $k_0 \in \mathbb{N}$ such that $|F^{k_0}(y') - G^{k_0}(x_0')| \geq \varepsilon$. For this k_0,

$$\begin{aligned}|F^{k_0}(y') - G^{k_0}(x_0')| &= |F(F^{k_0-1}(y')) \\ &\qquad - H \circ F(G^{k_0-1}(x_0'))| \\ &\leq |F(F^{k_0-1}(y')) \\ &\qquad - F(G^{k_0-1}(x_0'))| \\ &\qquad + |F(G^{k_0-1}(x_0')) \\ &\qquad - H \circ F(G^{k_0-1}(x_0'))| \\ &< \frac{1}{4} + \delta' < \frac{1}{3}\end{aligned}$$

where H is the lift of h with $H(x_0') \in [0, 1)$. On the other hand,

$$\begin{aligned}|F^k(y') - G^k(x_0')| &= d(\pi(F^k(y')), \pi(G^k(x_0'))) \\ &= d(f^{k_0}(y_0), g^{k_0}(x_0)) < \varepsilon.\end{aligned}$$

The contradiction shows that

$$|F^k(y') - G^k(x_0')| < \varepsilon$$

for all $k \in \mathbb{N}$. Since $\pi G^N(x_0') = g^N(x_0) = x_0 = \pi(x_0')$, it follows that $G^N(x_0') = x_0' + q$ for some $q \in \mathbb{N}$. Consequently,

$$\mu(f) = \lim_{k->\infty} \frac{F^k(y')}{k} = \lim_{k->\infty} \frac{G^k(x_0')}{k} = \frac{q}{N},$$

and the rotation number of f is rational, contradicting the initial assumption and so f must have periodic points. Thus, it may be assumed that $\mathrm{Fix}(f) \neq \emptyset$. Moreover, it is easy to check that if the set $\mathrm{Fix}(f)$ consists of one point then f does not have the shadowing property with respect to the class $\mathcal{T}_h$.

Next, if $(a, b) \cap \mathrm{Fix}(f) \neq \emptyset$ for $a, b \in \mathrm{Fix}(f)$ then the function $F(t) - t$ changes sign on (a, b), where F is the lift of f with $F(0) \in [0, 1)$. To see this, suppose $F(t) - t \geq 0$ for all $t \in (a, b)$, and let $c \in (a, b) \cap \mathrm{Fix}(f)$. For any $\gamma > 0$, there exists $g_\gamma \in Z(S)$ such that

$$d_0(f, g_\gamma) < \gamma \quad \text{and} \quad G(t) - t > 0 \text{ for } t \in (a, b),$$

where G is the lift of g_γ with $G(0) \in [0, 1)$. However, then no f-orbit in (a, b) can ϵ-shadow the g-orbit through c, and this contradiction shows that the function $F(t) - t$ changes sign on (a, b). Application of Lemma 2.6 completes the proof. ■

Theorem 2.7. *A homeomorphism f on a compact manifold M has the shadowing property with respect to the class $\mathcal{T}_0$ if and only if f has the shadowing property with respect to the class $\mathcal{T}_h$.*

Proof. Because of Theorem 2.5, it may assumed that $\dim M \geq 2$. Suppose that $f \in Z(M)$ has the shadowing property with respect to the class $\mathcal{T}_h$. By Lemma 1.1.1 of [Pilyugin, 1999], it suffices to show that f has the finite POTP.

To show this, let $\varepsilon > 0$ be arbitrary and choose $\delta_1 > 0$ corresponding to $\varepsilon/2$ in the definition of shadowing of f with respect to the class $\mathcal{T}_h$. Put $\delta = 1/4\delta_1$ and consider a finite δ-pseudo orbit $\xi = \{x_0, x_1, \dots, x_m\}$ of f. Then we can select a set $\xi' = \{y_0, y_1, \dots, y_m\}$ of points in M such that

(i) $d(x_k, y_k) < \delta_1$, $k = 0, 1, \dots, m$;
(ii) $d(f(y_k), y_{k+1}) < 2\delta$, $k = 0, 1, \dots, m-1$;
(iii) $y_i \neq y_j$, $0 \leq i < j \leq m$.

From Lemma 13 of [Nitecki & Shub, 1975], there is a homeomorphism $\phi \in Z(M)$ satisfying

$$d_0(\phi, 1_M) < \delta_1 \quad \text{and} \quad \phi(f(y_k)) = y_{k+1}$$

for $k = 0, 1, \dots, m-1$. Since $d_0(f, g) < \delta_1$, there exists a point $z \in M$ satisfying

$$d(f^k(z), g^k(y_0)) < \frac{\varepsilon}{2}$$

for $k = 0, 1, \dots, m$. Then,

$$d(f^k(z), x_k) < \epsilon$$

for all $k = 0, 1, \dots, m$. This completes the proof.

■

A homeomorphism $f \in Z(M)$ is said to be *topologically stable* if, given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $g \in Z(M)$ with $d_0(f, g) < \delta$ there is a continuous map $h : M \to M$ such that

$$d_0(h, 1_M) < \varepsilon \quad \text{and} \quad h \circ g = f \circ h.$$

A subset $Y \subset X$ is called *residual* if Y contains a countable intersection of open and dense subsets of X. If P is a property of elements of X, we say that this property is *generic* if the set $\{x \in X : x \text{ satisfies } P\}$ is residual.

It should be emphasized that topological stability is a much stronger property than the POTP.

It is well known that the topological stability is not a generic property. However, recently, Pilyugin and Plameneskaya [1999] showed that the POTP is generic.

The remainder of this section characterizes shadowing in terms of stability.

Definition 2.8. A homeomorphism $f \in Z(M)$ is said to be *perturbation stable* if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $g \in Z(M)$, with $d_0(f, g) < \delta$, there is a map $h : M \to M$ such that

$$d_0(h, 1_M) < \varepsilon \quad \text{and} \quad h \circ g = f \circ h\,.$$

Note that f is topologically stable if the map h is continuous.

Theorem 2.9. *A homeomorphism $f \in Z(M)$ has the POTP if and only if f is perturbation stable.*

Proof. Suppose that f has the POTP and let $\varepsilon > 0$ be arbitrary. Choose a $\delta > 0$ corresponding to this ε in the definition of POTP for f. Select $g \in Z(M)$ with $d_0(f, g) < \delta$. Let $O(g, x) = \{g^k(x) : k \in \mathbb{Z}\}$ for $x \in M$, and put $M/g = \{O(g, x) : x \in M\}$. Let $\alpha : M/g \to M$ be a choice function, and denote

$$\alpha(O(g, x)) = \overline{x},\ x \in M\,.$$

For each $\overline{x} \in M$ define the set

$$\vartheta(\overline{x}) = \{y \in M : d(f^k(y), g^k(\overline{x})) < \varepsilon \text{ for all } k \in \mathbb{Z}\}\,.$$

Then each $\vartheta(\overline{x})$ is nonempty. Let $\beta : \{\vartheta(\overline{x}) : x \in M\} \to M$ be another choice function and denote

$$\beta(\vartheta(\overline{x})) = \overline{y},\ x \in M\,.$$

Define a map $h : M \to M$ by

$$h(g^k(\overline{x})) = f^k(\overline{y})$$

for $x \in M$ and all $k \in \mathbb{Z}$. Then h is well defined and satisfies

$$d_0(h, 1_M) < \varepsilon \quad \text{and} \quad h \circ g = f \circ h\,.$$

To show these properties, let $\alpha(O(g, x_0)) = \overline{x}_0$ for each $x_0 \in M$. Then there exists $n \in \mathbb{Z}$ with $g^n(\overline{x}_0) = x_0$. Let $\beta(\vartheta(\overline{x}_0)) = \overline{y}_0$. Then

$$d(h(g^k(\overline{x}_0)), g^k(\overline{x}_0)) = d(f^k(\overline{y}_0), g^k(\overline{x}_0)) < \varepsilon$$

for all $k \in \mathbb{Z}$. This means that $d_0(h, 1_M) < \varepsilon$. Moreover,

$$\begin{aligned} hg(x_0) &= hg(g^n(\overline{x}_0)) = f^{n+1}(\overline{y}_0) \\ &= f(hg^n(\overline{x}_0)) = fh(x_0)\,. \end{aligned}$$

Conversely, it is clear that if f is perturbation stable then it has the shadowing property with respect to the class $\mathcal{T}_h$. Applying Theorem 2.7, it is easy to see that f has the shadowing property with respect to the class $\mathcal{T}_0$. The proof is complete. ■

3. Inverse Shadowing and Hyperbolicity

Many attempts have been made to express the concept of hyperbolicity in topological terms. Notions of shadowing, coordinates, expansiveness and the like have proven to be useful for this purpose. In this section concepts of hyperbolicity are characterized in terms of inverse shadowing, as introduced in [Corless & Pilyugin, 1995; Diamond *et al.*, 1993, 1995; Kloeden & Ombach, 1997].

When discussing inverse shadowing of $f \in Z(M)$, an appropriate choice of the class of admissible pseudo orbits is crucial [Corless & Pilyugin, 1995; Kloeden & Ombach, 1997; Lee & Choi, 2002], whereas the notion of shadowing of $f \in Z(M)$ is independent of the choice of the class of admissible pseudo orbits, as, for example, in Theorem 2.7.

Clearly, the inverse shadowing property with respect to the class $\mathcal{T}_0$ is stronger than that of inverse shadowing with respect to the class $\mathcal{T}_c$, and inverse shadowing with respect to the class $\mathcal{T}_c$ is stronger than that of inverse shadowing with respect to the class $\mathcal{T}_h$.

Throughout this section, the term "inverse shadowing property" refers to inverse shadowing with respect to the class $\mathcal{T}_h$, which is the weakest notion among the three.

Briefly, recall the concept of hyperbolicity. Let $f \in \mathrm{Diff}(M)$. Denote by T_xM the tangent space of M at $x \in M$, and let $\|v\|$ be the norm of $v \in T_xM$ induced by the Riemannian metric on M. Fix $x \in M$ and define two linear subspaces of T_xM:

$$\begin{aligned} E_x^s &= \{v \in T_xM : \|Df_x^n(v)\| \to 0 \text{ as } n \to \infty\} \\ E_x^u &= \{v \in T_xM : \|Df_x^{-n}(v)\| \to 0 \text{ as } n \to \infty\}\,, \end{aligned}$$

where Df denotes the derivative of f. A closed invariant set $\Lambda \subset M$ is said to be *hyperbolic* for f if each tangent space T_xM, $x \in \Lambda$, has a continuous direct sum $T_xM = E_x^s \oplus E_x^u$ whose summands are invariant under the map Df_x,

$$Df_x(E_x^s) = E_{f(x)}^s \quad \text{and} \quad Df_x(E_x^u) = E_{f(x)}^u .$$

If M is hyperbolic for f then f is called *Anosov.*

Hyperbolicity is a principal object of interest in the global qualitative theory of dynamical systems. It is well known that if $f \in \mathrm{Diff}(M)$ is Anosov then it is expansive and *structurally stable.* That is, given $\varepsilon > 0$ there exists a C^1 neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$ there is a homeomorphism $h : M \to M$ such that

$$d_0(h, 1_M) < \varepsilon \quad \text{and} \quad h \circ f = g \circ h .$$

Moreover, if $f \in \mathrm{Diff}(M)$ is structurally stable and expansive then it is Anosov [Mane, 1977].

A map $f \in Z(X)$ is *expansive* on a subset Y of X if there is $e > 0$, such that for any pair of distinct points $x \in X$ and $y \in Y$,

$$\sup_{k \in \mathbb{Z}} d(f^k(x), f^k(y)) > e ,$$

when $X = Y$ f is termed expansive, and $e > 0$ is called an *expansive constant* of f. For any $f \in Z(X)$ and $Y \subset X$, define the finite quantity

$$e_f(Y) = \sup\{e \geq 0 : d(f^k(x), f^k(y)) \leq e, \forall x \in X, \\ y \in Y \text{ and } \forall k \in \mathbb{Z} \text{ implies } x = y\}$$

and observe that $e_f(Y)$ is nonzero if and only if f is expansive on Y.

It has been shown that an expansive homeomorphism $f \in Z(X)$ with the shadowing property has the inverse shadowing property [Kloeden & Ombach, 1997, Theorem 1]. However an expansive homeomorphism $f \in Z(X)$ with the inverse shadowing property need not have the shadowing property. In fact, the pseudo Anosov diffeomorphism on a compact surface is expansive and has the inverse shadowing property [Lewowicz, 1983, Corollary 3.1], but does not have the POTP.

To obtain a necessary condition for the inverse shadowing property to imply the shadowing property, we introduce the notion of stable closed expansivity, which is slightly stronger than that of expansivity.

Definition 3.1. A map $f \in Z(X)$ is said to be *closed expansive* (*CLE*) if whenever f is expansive on a subset Y of X, f is also expansive on the closure $\overline{Y}$, and $e_f(\overline{Y}) = e_f(Y)$.

Remark 3.2. Clearly, if $f \in Z(X)$ has the shadowing property on a subset Y of X, then f has the shadowing property on the closure $\overline{Y}$. Similarly, if f has the inverse shadowing property on Y then f has the inverse shadowing property on the closure $\overline{Y}$. However, even if f is expansive on Y it cannot be guaranteed that f is expansive on the closure $\overline{Y}$.

If $f \in Z(X)$ is expansive then it is clearly CLE, but the converse does not hold. To illustrate this, for each $n \in \mathbb{N}$, let

$$X_n = \left\{ \frac{1}{2^n}\left(2 - \frac{1}{k}\right), \frac{1}{2^n}\left(1 + \frac{1}{k}\right) : k \in \mathbb{N} \right\},$$

and let $X = (\bigcup_{n \in \mathbb{N}} X_n) \bigcup \{0\}$. Then X is a compact subset of $[0, 1]$. Define a homeomorphism f on X by

$$f(x) = \begin{cases} x & \text{if } x \in \left\{ \dfrac{1}{2^k} : k = 0, 1, 2, \ldots \right\} \cup \{0\}, \\ x' & \text{otherwise,} \end{cases}$$

where x' is the greatest element of X which is smaller than x. Then f is CLE, but it is not expansive. Moreover, it is easy to construct an expansive homeomorphism $f \in Z(X)$ such that:

1. f has a neighborhood $\mathcal{U}$ such that every $g \in \mathcal{U}$ is CLE; but
2. any neighborhood $\mathcal{V}$ of f contains a homeomorphism $h \in \mathcal{U}$ which is not expansive.

In fact, in the above example, the restriction map $f|_{X_1} : X_1 \to X_1$ is such a homeomorphism.

Definition 3.3. The map $f \in \mathrm{Diff}(M)$ is said to be *stably closed expansive* (*SCLE*) if it is expansive and there exists a C^1 neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}(M)$ such that every $g \in \mathcal{U}$ is CLE.

Theorem 3.4. *A map $f \in \mathrm{Diff}(M)$ is Anosov if and only if it is SCLE and has the inverse shadowing property.*

Proof. Suppose that f is Anosov. Since the C^1 Anosov diffeomorphisms form an open subset of $\mathrm{Diff}(M)$, it is clear that f is SCLE. Moreover f has the inverse shadowing property, because it is expansive and has the shadowing property.

Conversely, suppose that f is SCLE and has the inverse shadowing property. It suffices to show that f is structurally stable. Let $e > 0$ be an expansive constant of f and $\varepsilon > 0$ a constant with $\varepsilon < e/24$. Choose $\delta > 0$ corresponding to this $\varepsilon > 0$ in the inverse shadowing property and the SCLE definition for f. Let $g \in \mathrm{Diff}(M)$ be such that $d_1(f, g) < \delta$, where d_1 is the metric on $\mathrm{Diff}(M)$ which induces the C^1 topology on $\mathrm{Diff}(M)$. Put $M/f = \{O(f, x) : x \in M\}$. Let $\alpha : M/f \to M$ be a choice function and write

$$\alpha(O(f, x)) = \overline{x},\ x \in M\,.$$

For each $\overline{x} \in M$, define the set

$$\Xi(\overline{x}) = \{y \in M : d(f^k(\overline{x}),\ g^k(y)) < \varepsilon \text{ for all } k \in \mathbb{Z}\}\,.$$

Then each $\Xi(\overline{x})$ is a nonempty set. Let $\beta : \{\Xi(\overline{x}) : x \in M\} \to M$ be another choice function, and write

$$\beta(\Xi(\overline{x})) = \overline{y},\ x \in M\,.$$

Define the map $h : M \to M$ by

$$h(f^k(\overline{x})) = g^k(\overline{y})$$

for $x \in M$ and all $k \in \mathbb{Z}$. Then h is a well defined injective map satisfying

$$d_0(h, 1_M) < \varepsilon \quad \text{and} \quad h \circ f = g \circ h\,.$$

To show this, since $\alpha(O(f, x_0)) = \overline{x}_0$ for each $x_0 \in M$, there exists some $n \in \mathbb{Z}$ with $f^n(\overline{x}_0) = x_0$. Let $\beta(\Xi(\overline{x}_0)) = \overline{y}_0$. Then

$$d(h(f^k(\overline{x}_0)),\ f^k(\overline{x}_0)) = d(g^k(\overline{y}_0),\ f^k(\overline{x}_0)) < \varepsilon$$

for all $k \in \mathbb{Z}$. Thus, $d_0(h, 1_M) < \varepsilon$. Moreover,

$$\begin{aligned} hf(x_0) &= hf(f^n(\overline{x}_0)) = g^{n+1}(\overline{y}_0) \\ &= g(hf^n(\overline{x}_0)) = gh(x_0). \end{aligned}$$

To show that h is injective, suppose that $h(x) = h(x_0)$ for $x, x_0 \in M$. Since $h \circ f = g \circ h$,

$$\begin{aligned} d(f^k(x),\ f^k(x_0)) &\le d(f^k(x),\ hf^k(x)) \\ &\quad + d(hf^k(x_0),\ f^k(x_0)) < e\,, \end{aligned}$$

for all $k \in \mathbb{Z}$, giving $x = x_0$ by expansivity.

Next, the map h is continuous. First, the map g is expansive on the set $h(M)$ with an expansive constant $e/2$. To show this, suppose that

$$d(g^k(h(x)),\ g^k(h(y))) < \frac{e}{2}$$

for $x, y \in M$ and all $k \in \mathbb{Z}$. Then

$$\begin{aligned} d(f^k(x),\ f^k(y)) &\le d(f^k(x),\ g^k(h(x))) \\ &\quad + d(g^k(h(x)),\ g^k(h(y))) \\ &\quad + d(g^k(h(y)),\ f^k(y)) < e \end{aligned}$$

for all $k \in \mathbb{Z}$, and so $h(x) = h(y)$. Let $\lambda > 0$ be arbitrary. Then there is $n \ge 1$ such that

$$d(g^k(h(x)),\ g^k(h(y))) \le \frac{e}{3} \quad \text{for all } |k| < n \quad \text{implies } d(h(x),\ h(y)) < \lambda,$$

for $x, y \in M$. Suppose the contrary. Then there exists $\gamma > 0$ such that for each $n \ge 1$ there are two points $x_n, y_n \in M$ satisfying

$$d(g^k(h(x_n)),\ g^k(h(y_n))) \le \frac{e}{3}$$

and

$$d(h(x_n),\ h(y_n)) \ge \gamma$$

for all $|k| \le n$. Since M is compact, without loss of generality it may be assumed that the sequences $\{h(x_n)\}$ and $\{h(y_n)\}$ converge, say $h(x_n) \to z$ and $h(y_n) \to w$. Then $z, w \in \overline{h(M)}$ and $d(z, w) \ge \gamma$. Since g is CLE and

$$\begin{aligned} d(g^k(z),\ g^k(w)) &\le d(g^k(z),\ g^k(h(x_n))) \\ &\quad + d(g^k(h(x_n)),\ g^k(h(y_n))) \\ &\quad + d(g^k(h(y_n)),\ g^k(w)) < \frac{e}{2}\,, \end{aligned}$$

for sufficiently large n and for all $k \in \mathbb{Z}$, we arrive at a contradiction. Now let $\eta > 0$ be such that if $d(x, y) < \eta$ then $d(f^k(x),\ f^k(y)) < e/6$ for all $|k| \le n$. Then

$$d(g^k(h(x)),\ g^k(h(y))) < \frac{e}{3}$$

for all $|k| < n$ and so $d(h(x), h(y)) < \lambda$.

It is well known that if M is a compact manifold and $d_0(h, 1_M) < \varepsilon$ with ε small enough, then h maps M onto M. This means that f is structurally stable, and the proof is complete. ■

Corollary 3.5. *Let $f \in \mathrm{Diff}(M)$ be SCLE. Then f has the shadowing property if and only if it has the inverse shadowing property.*

Remark 3.6. If $f \in \mathrm{Diff}(M)$ is Anosov then it has the inverse shadowing property. Furthermore, it is easy to see that the inverse shadowing point remains valid under a small C^1 perturbation of f. In fact, if f is Anosov a common expansive constant $e > 0$ can be found for all diffeomorphisms in some C^1 neighborhood $\mathcal{U}_1$ of f. Choose a C^1 neighborhood $\mathcal{U}_2$ of f corresponding to the number $e/2$ in the inverse shadowing property of f. Let $\mathcal{U} = \mathcal{U}_1 \bigcap \mathcal{U}_2$ and $g \in \mathcal{U}$. Then every f-orbit is e-shadowed by a unique g-orbit.

Definition 3.7. Say that $f \in \mathrm{Diff}(M)$ has the *inverse shadowing uniqueness property* (resp. *weak inverse shadowing uniqueness property*) if there exists $L > 0$ such that for any $\varepsilon > 0$ with $\varepsilon < L$ there is a C^0 neighborhood (resp. C^1 neighborhood) $\mathcal{U}$ of f in $\mathrm{Diff}(M)$, such that for any $g \in \mathcal{U}$ and any $x \in M$ there is a unique point $y \in M$ for which

$$d(f^k(x), g^k(y)) < \varepsilon$$

for all $k \in \mathbb{Z}$.

Theorem 3.8. *A map $f \in \mathrm{Diff}(M)$ is Anosov if and only if f has the weak inverse shadowing uniqueness property.*

Proof. By Remark 3.6, it is enough to show the sufficiency. Suppose that f has the weak inverse shadowing uniqueness property. Then it is clear that f is expansive. Let $e > 0$ be an expansive constant of f and $\varepsilon > 0$ a constant with $\varepsilon < e/12$. Choose $\delta > 0$ corresponding to this $\varepsilon > 0$ in Definition 3.7. Let $g \in \mathrm{Diff}(M)$ be such that $d_1(f, g) < \delta$, and let $x \in M$. Then there exists a unique point $h(x)$ whose g-orbit ε-shadows $\{f^k(x) : k \in \mathbb{Z}\}$. This defines a map $h : M \to M$ with

$$d_0(h, 1_M) < \varepsilon \quad \text{and} \quad h \circ f = g \circ h .$$

In fact, for any $x \in M$,

$$d(f^k(f(x)), g^k(hf(x))) < \varepsilon$$

and

$$d(f^{k+1}(x), g^{k+1}(h(x))) < \varepsilon$$

for all $k \in \mathbb{Z}$. By the uniqueness, $h(f(x)) = g(h(x))$. To show that h is injective, suppose that $h(x) = h(y)$ for $x, y \in M$. Then

$$d(f^k(x), g^k(h(x)) < \varepsilon \text{ and } d(f^k(y), g^k(h(y))) < \varepsilon$$

for all $k \in \mathbb{Z}$, giving $x = y$.

Next, the map h is continuous. It can be easily shown that the map g is expansive on the set $h(M)$ with an expansive constant $e/4$; but it need not be expansive on the set $\overline{h(M)}$. Let $\lambda > 0$ be arbitrary. Then there exists $n \geq 1$ such that

$$d(g^k(x), g^k(y)) \leq \frac{e}{4} \quad \text{for all } |k| < n$$
$$\text{implies } d(x, y) < \lambda ,$$

for $x, y \in h(M)$. For, if not there exists $\mu > 0$ such that for each $n \geq 1$ there are two points $x_n, y_n \in M$ satisfying

$$d(g^k(h(x_n)), g^k(h(y_n))) \leq \frac{e}{4}$$

and

$$d(h(x_n), h(y_n)) \geq \mu$$

for all $|k| \leq n$. Since M is compact, there exist subsequences $\{x_{n_i}\}, \{y_{n_i}\}$ of $\{x_n\}$ and $\{y_n\}$, respectively, such that

$$x_{n_i} \to x \quad \text{and} \quad y_{n_i} \to y .$$

Choose a subsequence $\{n_{i_j}\}$ of $\{n_i\}$ such that

$$h(x_{n_{i_j}}) \to z \quad \text{and} \quad h(y_{n_{i_j}}) \to w .$$

Then for any given $k > 0$ obtain that

$$\begin{aligned} d(f^k(x), g^k(z)) \leq d(f^k(x), f^k(x_{n_{i_j}})) \\ + d(f^k(x_{n_{i_j}}), g^k(h(x_{n_{i_j}}))) \\ + d(g^k(h(x_{n_{i_j}})), g^k(z)) \leq \varepsilon \end{aligned}$$

for sufficiently large j. This means that $h(x) = z$, and so

$$h(x_{n_{i_j}}) \to h(x) .$$

Similarly,

$$h(y_{n_{i_j}}) \to h(y) .$$

Consequently,

$$d(g^k(h(x)), g^k(h(y))) \leq \frac{e}{4}$$

and

$$d(h(x), h(y)) \geq \mu$$

for all $k \in \mathbb{Z}$. This contradicts the fact that g is expansive on $h(M)$ with an expansive constant $e/4$. Choose $\eta > 0$ such that if $d(x, y) < \eta$ then $d(f^k(x), f^k(y)) < e/6$ for all $|k| < n$. Thus, $d(h(x), h(y)) < \lambda$ and the map h is continuous. By much the same reason as at the end of the proof of Theorem 3.4, it follows that f is Anosov. ■

Remark 3.9. There is no such result as the above for the (weak) shadowing uniqueness property. Say that $f \in \mathrm{Diff}(M)$ has the *shadowing uniqueness property* (resp. *weak shadowing uniqueness property*) if there exist $L > 0$ such that for any $\varepsilon > 0$ with $\varepsilon < L$ there is a C^0 neighborhood (resp. C^1 neighborhood) $\mathcal{U}$ of f in $\mathrm{Diff}(M)$ such that for any $g \in \mathcal{U}$ and any $y \in M$ there is a unique point $x \in M$ for which $d(f^k(x), g^k(y)) < \varepsilon$ for all $k \in \mathbb{Z}$. It can be easily checked that if $f \in \mathrm{Diff}(M)$ is Anosov then it has the shadowing uniqueness property, but the converse does not hold. In fact, every diffeomorphism $g \in \mathrm{Diff}(M)$ which is topologically conjugate to a diffeomorphism f with the shadowing uniqueness property has the shadowing uniqueness property. But a diffeomorphism $g \in \mathrm{Diff}(M)$ which is topologically conjugate to an Anosov diffeomorphism f need not be Anosov.

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