

ATTRACTORS UNDER DISCRETIZATIONS WITH VARIABLE STEPSIZE

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ABSTRACT. The standard upper and lower semicontinuity results for discretized attractors [22], [13], [5] are generalized for discretizations with variable stepsize. Several examples demonstrate that the limiting behaviour depends crucially on the stepsize sequence. For stepsize sequences suitably chosen, convergence to the exact attractor in the Hausdorff metric is proven. Connections to pullback attractors in cocycle dynamics are pointed out.

1. Introduction. Attractors belong to the most important objects of phase space dynamics. Starting from the pioneering paper by Kloeden and Lorenz [16], (compact) attractors play an eminent role in numerical dynamics, the qualitative theory of discretization methods, too. Dozens of references can be found in the monographs of Stuart and Humphries [22], Grüne [13], and Cheban [5]. Though the development of numerical dynamics has been parallel for ordinary and for other types of differential equations and involves more general types of invariant sets [22], [13], [5], [9] than attractors, we focus our attention to (compact) attractors and remain within the framework (of one-step discretization methods) of ordinary differential equations.

As it is discussed in Chapter 7 of [22], there are four basic results on the constant stepsize discretized dynamics near a compact attractor \mathcal{A} :

- (A): For stepsize h sufficiently small, there exists a discretized attractor \mathcal{A}_h ;
- (B): $\mathcal{A}_h \rightarrow \mathcal{A}$ in an upper semicontinuous way;
- (C): $\mathcal{A}_h \rightarrow \mathcal{A}$ in a lower semicontinuous way provided that the family $\{\mathcal{A}_h\}$ is uniformly exponentially attracting;
- (D): $\mathcal{A}_h \rightarrow \mathcal{A}$ in a lower semicontinuous way provided that \mathcal{A} is the closure of a finite union of unstable manifolds of hyperbolic equilibria.

The result in (C) has been recently generalized by Grüne [11], [12], [13] to

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(\mathcal{C}^*): $\mathcal{A}_h \rightarrow \mathcal{A}$ in a lower semicontinuous way if and only if the family $\{\mathcal{A}_h\}$ has a uniform rate of attraction.

Though the constant stepsize attractor \mathcal{A}_h has no genuine counterpart in the variable stepsize case (see Example 1), results of type (B), (\mathcal{C}^*) make sense and constitute the content of Section 2 of the present paper. Section 1 is of introductory character. (Results of type (D) belong to the theory of discretizing invariant manifolds. We do not discuss them in this paper.)

Variable stepsize dynamics is a part of nonautonomous dynamics. This is particularly transparent in Remark 2 where connections to pullback attractors of cocycle dynamics are investigated and an existence result of Kloeden and Schmalfuss [17] — via dropping assumption $\liminf_m h_m > 0$ — is generalized.

In contrast to the frequent use of variable timesteps in computing practice, the number of results on variable stepsize discretizations is rather limited. The recent papers by Lamba and Stuart [21], [19], [20] have a distinguished status among them. Under some conditions on the underlying dynamics (excluding certain types of equilibria), they provide theoretical justification for a class of stepsize selection algorithms including a Runge–Kutta MATLAB implementation. The last paper of the series [20] contains an upper semicontinuity result for approximating attractors (similar to our Theorem 1 below) within this framework.

Numerical dynamics with variable stepsize can be considered as part of a future theory of comparing dynamical systems on different time scales. Though time scale dynamics (i.e. dynamic equations with time being a closed subset of the real line arbitrarily chosen) is rather developed by now [4], complicated questions of comparing dynamical systems on different time scales are not studied as yet.

1.1. Variable stepsize discretizations. We consider an autonomous differential equation

$$\dot{x} = f(x) \quad , \quad x \in \mathbb{R}^n \quad (1)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is of class C^{p+k+1} for some integers $p \geq 1$, $k \geq 0$. We consider also a p -th order C^{p+k+1} discretization operator $\varphi : [0, h_0] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ where h_0 is a positive constant. We make the technical assumption that, up to the order $p + k + 1$, the (mixed) partial derivatives of f and φ are (uniformly, on the whole \mathbb{R}^n resp. $[0, h_0] \times \mathbb{R}^n$) bounded. If our interest is focused on a compact subset of \mathbb{R}^n , then the fulfilment of these boundedness assumptions (e.g. by modifying f outside a large ball and assuming that φ comes from a general r -stage explicit or implicit Runge–Kutta method) can be taken for granted. We require also that φ is locally determined in the sense that, for some continuous function $\Delta : [0, h_0] \rightarrow \mathbb{R}^+$ with $\Delta(0) = 0$, $\varphi(h, x)$ depends only on the restriction of f to the set $\{y \in \mathbb{R}^n : |y - x| \leq \Delta(h)\}$.

Our assumptions imply that nonextendable solutions of (1) define a C^{p+k+1} continuous-time dynamical system $\Phi : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, the solution flow of (1). For later purposes, we note that

$$|\Phi(t, x) - \Phi(\tilde{t}, x)| \leq \gamma |t - \tilde{t}| \quad \text{for each } t, \tilde{t} \in \mathbb{R} \text{ and } x \in \mathbb{R}^n \quad (2)$$

and, as a direct consequence of the Gronwall inequality,

$$|\Phi(t, x) - \Phi(t, \tilde{x})| \leq e^{\Gamma|t|} |x - \tilde{x}| \quad \text{for each } t \in \mathbb{R} \text{ and } x, \tilde{x} \in \mathbb{R}^n \quad (3)$$

where

$$\gamma = \sup\{|f(x)| : x \in \mathbb{R}^n\} \quad \text{and} \quad \Gamma = \sup\{|f_x(x)| : x \in \mathbb{R}^n\}.$$

Similarly, for h sufficiently small, say $h \leq h_0$, the discretization method $\varphi(h, \cdot)$ is a C^{p+k+1} self-diffeomorphism of \mathbb{R}^n and the iterates of $\varphi(h, \cdot)$ form a discrete-time dynamical system on \mathbb{R}^n .

Given $x \in \mathbb{R}^n$ and a real sequence h_1, h_2, \dots with $0 \leq h_m \leq h_0$, we define inductively that

$$\varphi(h_{m+1}, \dots, h_1; x) = \varphi(h_{m+1}, \varphi(h_m, \dots, h_1; x)) , \quad m = 1, 2, \dots .$$

The collection of mappings $\{\varphi(h_m, \dots, h_1; \cdot)\}_{m=1}^\infty$ is said to be the induced *discretization process with variable stepsizes* h_1, h_2, \dots . Note that the starting timestep is h_1 whereas h_0 is simply an upper bound for $\sup_{m=1}^\infty h_m$. For simplicity, we assume that $h_0 < 1$.

It is not hard to give estimates for the difference between exact and approximating solutions on finite time intervals. Though standard numerical dynamics requires only C^0 and C^1 estimates, also the case of higher order derivatives (with respect to x) is incorporated.

Lemma 1. ERROR ESTIMATES ON INTERVAL $[0, T]$ *Given $T > 0$ arbitrarily, there exists a positive constant $K_0(T)$ such that*

$$|\Phi(h_M + \dots + h_1, \cdot) - \varphi(h_m, \dots, h_1; \cdot)| \leq K_0(T) \cdot h^p \tag{4}$$

whenever $h_M + \dots + h_1 \leq T$, $M = 1, 2, \dots$. It is well-known that $K_0(T)$ can be chosen for $\text{const}_0 \cdot T e^{\Gamma T}$. The corresponding C^j ($j = 1, 2, \dots, p + k + 1$) estimates are

$$|\Phi^{(j)}(h_M + \dots + h_1, \cdot) - \varphi^{(j)}(h_m, \dots, h_1; \cdot)| \leq K_j(T) \cdot h_0^{\mu(j)} \tag{5}$$

where $\mu(j) = \max\{p, p+k-j\}$ and $K_j(T)$ can be chosen for $\text{const}_j \cdot (T+T^j)e^{\Gamma(j+1)T}$, $j = 1, 2, \dots, p + k + 1$.

The proof of (5) goes by induction on j and is just a little harder than the proof of the corresponding result for constant stepsize sequences [8]. The basic idea is to use telescope summation. The emerging lower approximating sums on the nonuniform mesh have to be replaced by the respective integrals.

The abstract definition of discretization operators in the first paragraph of Subsection 1.1 goes back to [2]. It is slightly more restricted than the one used in [22].

1.2. Semicontinuous convergence of set sequences. Let (X, d) be a metric space. The collection of nonempty compact subsets of X is denoted by $\mathcal{C}_H(X)$. The Hausdorff metric on $\mathcal{C}_H(X)$ is defined by letting

$$d_H(P, Q) = \max\{\text{asd}(P, Q), \text{asd}(Q, P)\} \quad \text{for each } P, Q \in \mathcal{C}_H(X)$$

where $\text{asd}(P, Q)$ stays for the asymmetric semidistance $\max\{d(p, Q) \mid p \in P\}$ from the set P to the set Q , and $d(p, Q) = \min\{d(p, q) \mid q \in Q\}$, the distance between the point p and the set Q . It is well-known that $(\mathcal{C}_H(X), d_H)$ is compact if (X, d) is compact.

From now on, consider a sequence $\{S_k\}_k \subset \mathcal{C}_H(X)$ and an $S \in \mathcal{C}_H(X)$. Given $\varepsilon > 0$ arbitrarily, we set

$$B(S, \varepsilon) = \{x \in X \mid d(x, S) < \varepsilon\} ,$$

the open ε -neighborhood of S in X .

Lemma 2. UPPER SEMICONTINUITY FOR SET SEQUENCES *The following statements are pairwise equivalent:*

- (i)_U: $S_k \rightarrow S$ in an upper semicontinuous way;
- (ii)_U: Given $\varepsilon > 0$ arbitrarily, there exists a $K(\varepsilon) \in \mathbb{N}$ such that $S_k \subset B(S, \varepsilon)$ for each $k > K(\varepsilon)$;
- (iii)_U: $asd(S_k, S) \rightarrow 0$.

If, in addition, $cl(\cup_k S_k)$ is compact, then (i)_U is equivalent to

- (iv)_U: $limsup_k S_k \subset S$ where

$$limsup_k S_k = \{x \in X \mid \text{given } \varepsilon > 0 \text{ arbitrarily, } B(x, \varepsilon) \cap S_k \neq \emptyset$$

$$\text{holds true for infinitely many indices } k\} = \bigcap_{K \geq 0} cl \left(\bigcup_{k \geq K} S_k \right).$$

Lemma 3. LOWER SEMICONTINUITY FOR SET SEQUENCES *The following statements are pairwise equivalent:*

- (i)_L: $S_k \rightarrow S$ in a lower semicontinuous way;
- (ii)_L: Given $\varepsilon > 0$ arbitrarily, there exists a $K(\varepsilon) \in \mathbb{N}$ such that $S \subset B(S_k, \varepsilon)$ for each $k > K(\varepsilon)$;
- (iii)_L: $asd(S, S_k) \rightarrow 0$.

If, in addition, $liminf_k S_k$ is compact, then (i)_L is equivalent to

- (iv)_L: $liminf_k S_k \supset S$ where

$$liminf_k S_k = \{x \in X \mid \text{given } \varepsilon > 0 \text{ arbitrarily, } B(x, \varepsilon) \cap S_k \neq \emptyset$$

$$\text{for all indices sufficiently large}\} = \bigcap_{\varepsilon > 0} \left(\bigcup_{K \geq 0} \left(\bigcap_{k \geq K} B(S_k, \varepsilon) \right) \right).$$

Note that simultaneous upper and lower semicontinuous convergence in $\mathcal{C}_H(X)$ is equivalent to convergence in the Hausdorff metric. For later purposes, recall that

$$limsup_k S_k = \{x \in X \mid \text{there exists an index sequence } n_k \text{ with } n_k \rightarrow \infty$$

$$\text{and a sequence of points } x_k \in S_{n_k} \text{ such that } x_k \rightarrow x\} \tag{6}$$

and

$$liminf_k S_k = \{x \in X \mid \text{there exists a sequence of points } x_k \in S_k$$

$$\text{with the property that } x_k \rightarrow x \text{ as } k \rightarrow \infty\}.$$

All results in Subsection 1.2 can be found e.g. in [1].

2. Exact and approximating attractors. With \mathbb{T} denoting \mathbb{R} or \mathbb{Z} , let $\pi : \mathbb{T} \times \mathbb{R}^n$ be a continuous-time or discrete-time dynamical system. A set S in \mathbb{R}^n is said to be invariant if $x \in S$ implies that $\pi(t, x) \in S$ for all $t \in \mathbb{T}$. A nonempty compact invariant set \mathcal{A} is said to be an *attractor* if it admits a neighborhood \mathcal{N} such that for each neighborhood U of \mathcal{A} there is a number $t_U \in \mathbb{R}$ with $\{\pi(t, x) \in \mathbb{R}^n : (t, x) \in \mathbb{T} \times \mathcal{N}, t \geq t_U\} \subset U$. The *region of attraction* is the (necessarily open and invariant) set

$$R(\mathcal{A}) = \{x \in \mathbb{R}^n \mid d(\pi(t, x), \mathcal{A}) \rightarrow 0 \text{ as } t \rightarrow \infty\}$$

where $d(\pi(t, x), \mathcal{A}) = \min\{|\pi(t, x) - a| : a \in \mathcal{A}\}$, the distance between the point $\pi(t, x)$ and the compact set \mathcal{A} . Obviously, $\mathcal{N} \subset R(\mathcal{A})$ and there is no loss of generality in assuming that \mathcal{N} is compact. Then $\mathcal{A} = \cap\{\pi(t, \mathcal{N}) : t \in \mathbb{T} \text{ and } t \geq 0\}$ and, as $t \rightarrow \infty$ in \mathbb{T} , $\pi(t, \mathcal{N}) \rightarrow \mathcal{A}$ in the Hausdorff metric.

Let \mathcal{A} be an attractor of Φ , the solution flow of (1). Starting from the pioneering work of Kloeden and Lorenz [16], a great number of papers have been devoted to the

question if attractors persist under discretization. The basic result is that, for h sufficiently small, the discrete-time dynamical system $\varphi(h, \cdot)$ has an attractor \mathcal{A}_h such that, with $h \rightarrow 0^+$, \mathcal{A}_h approaches \mathcal{A} in an upper semicontinuous way. The constant stepsize approximate attractor \mathcal{A}_h can be given as $\mathcal{A}_h = \cap\{\varphi^m(h, \mathcal{N}) \mid m \in \mathcal{N}\}$. Various extra conditions imply that the limiting process $\mathcal{A}_h \rightarrow \mathcal{A}$ is also lower semicontinuous. In line with $(ii)_U$ (resp. $(ii)_L$), upper (resp. lower) semicontinuous convergence means that, given $\varepsilon > 0$ arbitrarily, there exists an $h(\varepsilon) \in (0, h_0]$ with the property that $\mathcal{A}_h \subset B(\mathcal{A}, \varepsilon)$ (resp. $\mathcal{A} \subset B(\mathcal{A}_h, \varepsilon)$) whenever $h \in (0, h(\varepsilon)]$. For details, as well as for similar results on other types of evolution equations and discretization procedures, see [22], [13], [5]. The beautiful paper by Hill and Süli [14] is worth of being mentioned separately. It contains an abstract version of various earlier upper semicontinuity results on discretized compact attractors for semidynamical systems in Banach spaces (whereas local compactness of the phase space is replaced by a compactifying assumption on the dynamics). The presentation is particularly simple because of exploiting — in the theory of discretized attractors for the very first time — the concepts of limes superior and limes inferior for sets.

Here we recall only the basic result on the link between discretizations and the level surface structure of Liapunov functions on $R(\mathcal{A}) \setminus \mathcal{A}$. Starting from [16], several versions of this result are given in the literature [22], [13], [5]. Lemma 4 as stated below is taken from [10].

Lemma 4. DISCRETIZATION AND C^∞ LIAPUNOV FUNCTIONS *There exists a C^∞ and globally Lipschitz function $V : \mathbb{R}^n \rightarrow [0, 1]$ satisfying*

$$\begin{aligned} V(x) &= 0 && \text{if and only if } x \in \mathcal{A}, \\ V(x) &= 1 && \text{if and only if } x \in \mathbb{R}^n \setminus R(\mathcal{A}), \\ V(x) &\rightarrow 1 && \text{as } |x| \rightarrow \infty, \end{aligned}$$

and, last but not least,

$$\langle \text{grad } V(x), f(x) \rangle < 0 \quad \text{whenever } x \in R(\mathcal{A}) \setminus \mathcal{A}.$$

In addition, given $c \in (0, 1)$ arbitrarily, there exists an $h^* = h^*(c) > 0$ such that

$$V(\varphi(h, x)) < c \quad \text{for all } (h, x) \in (0, h^*] \times V^{-1}([0, c]).$$

Corollary 1. *Assume that $\mathcal{N} = V^{-1}([0, c_0])$ for some fixed $c_0 \in (0, 1)$. Then, given $\tau \in \mathbb{R}$ arbitrarily,*

$$\varphi(h, \Phi(\tau, \mathcal{N})) \subset \text{int}(\Phi(\tau, \mathcal{N})) \quad \text{for each } h \in (0, h_*(\tau)]$$

with some $h_*(\tau) > 0$.

Proof. This is just a version of the last assertion of Lemma 4. Details are left to the reader. Also the method outlined in the forthcoming Remark applies. QED

Remark 1. Assume that all conditions of Corollary 1 are satisfied. Then $\varphi(h, \mathcal{N}) \subset \text{int}(\mathcal{N})$ and, a fortiori, $\{\varphi^m(h, \mathcal{N})\}_{m=1}^\infty$ forms a nested sequence of compacta in \mathcal{N} , for each $h \in (0, h^*(c_0)]$. Moreover, for h, \tilde{h} sufficiently small, we have that $\varphi(\tilde{h}, \mathcal{N}) \subset \text{int}(\varphi(h, \mathcal{N}))$ whenever $\tilde{h} > h$. In fact, since $\varphi(h, \cdot)$ is a self-diffeomorphism of \mathbb{R}^n , it is sufficient to point out that

$$q_x(h) = V(\varphi^{-1}(h, \varphi(\tilde{h}, x))) < V(x) \quad \text{whenever } (h, x) \in (0, \tilde{h}) \times V^{-1}(c_0),$$

\tilde{h} sufficiently small. But $q_x(0) = V(\varphi(\tilde{h}, x)) < c_0$, $q_x(\tilde{h}) = V(x) = c_0$ and, as the result of a straightforward but rather lengthy computation, $\dot{q}_x(h) > 0$ for each $h \in [0, \tilde{h}]$, \tilde{h} sufficiently small (and not depending on x).

2.1. Stepsize choice and semicontinuity. Returning to variable stepsize sequences, we set (for a general \mathcal{N})

$$t_m = h_m + \cdots + h_1 \quad \text{and} \quad \mathcal{N}_{t_m} = \varphi(h_m, \dots, h_1; \mathcal{N}), \quad m = 1, 2, \dots$$

Assumptions $\mathcal{N} = V^{-1}([0, c_0])$ and $h_0 \leq h^*(c_0)$ imply that $\mathcal{N}_{t_m} \subset \mathcal{N}$ for each m . However, in contrast to the constant stepsize case, the sequence of compacta $\{\mathcal{N}_{t_m}\}_{m=1}^\infty$ is not necessarily nested and may have more than one accumulation point in the Hausdorff metric.

Example 1. Let $f : \mathbb{R} \rightarrow [-1, 1]$ be an odd C^∞ function with $f^{-1}(0) = [-1, 1]$ and $f^{-1}(1) = (-\infty, -2]$. It is clear that $\mathcal{A} = [-1, 1]$ is an attractor for the solution flow Φ of the differential equation $\dot{x} = f(x)$ and $R(\mathcal{A}) = \mathbb{R}$. In addition, let $\{c_i\}_{i=0}^\infty$ be a strictly decreasing real sequence with $c_0 = h_0$ and $c_i \rightarrow 0$ as $i \rightarrow \infty$. Set $I_i = (c_{i+1}, c_i)$, $i \in \mathbb{N}$. It is not hard to construct a C^∞ function $s : [0, h_0] \times \mathbb{R} \rightarrow [-1, 1]$ with the properties that

$$s^{-1}((0, 1]) = (\cup_{k \in \mathbb{N}} I_{2k}) \times (-\infty, 1) \quad , \quad s^{-1}([-1, 0)) = (\cup_{k \in \mathbb{N}} I_{2k+1}) \times (-1, \infty).$$

With some integer $p \geq 1$, define

$$\varphi(h, x) = \Phi(h, x) + s(h, x)h^{p+1} \quad \text{whenever} \quad (h, x) \in [0, h_0] \times \mathbb{R}.$$

By the construction, φ is a p -th order C^∞ discretization operator. Set $\mathcal{N} = [-2, 2]$ and consider a sequence $\{\tau_i\}_{i=0}^\infty$ with $\tau_i \in I_i$. Given a sequence of integers $0 = M(0) < M(1) < M(2) < \dots$, set

$$h_m = \tau_k \quad \text{whenever} \quad m = M(k) + 1, M(k) + 2, \dots, M(k+1) \quad \text{for some} \quad k \in \mathbb{N}.$$

It is readily seen that, with τ_k and $M(k)$ properly chosen, our discretization process with the variable stepsize sequence h_1, h_2, \dots satisfies

$$\mathcal{N}_{t_{M(2k+1)}} \subset \left[\frac{k}{k+1}, \frac{k+2}{k+1} \right] \quad \text{and} \quad \mathcal{N}_{t_{M(2k+2)}} \subset \left[\frac{-k-2}{k+1}, \frac{-k}{k+1} \right], \quad k \in \mathbb{N}.$$

In particular, $d_H(\mathcal{N}_{t_{M(2k+1)}}, \{1\}) \rightarrow 0$, $d_H(\mathcal{N}_{t_{M(2k+2)}}, \{-1\}) \rightarrow 0$ and, with some little more care, $\limsup_m \mathcal{N}_{t_m} = [-1, 1]$, $\liminf_m \mathcal{N}_{t_m} = \emptyset$.

In the spirit of the Kloeden–Lorenz Theorem [16] recalled at the top of the second paragraph of this Section, our next result concerns families of variable stepsize sequences and, conceptually, it is equivalent to the original result in [16]. Upper and lower semicontinuity for individual stepsize sequences are considered in Theorems ?? and 3 below.

Theorem 1. *Assume that \mathcal{A} is an attractor for (1) and let \mathcal{N} be a compact neighborhood of \mathcal{A} in $R(\mathcal{A})$. Given $\varepsilon > 0$ arbitrarily, there exists an $\eta > 0$ and a time $\tau > 0$ with the properties as follows. For any stepsize sequence h_1, h_2, \dots satisfying $h_0 < \eta$ and $\sum_m h_m = \infty$,*

$$\mathcal{N}_{t_m} \subset B(\mathcal{A}, \varepsilon) \quad \text{whenever} \quad t_m \geq \tau.$$

Proof. There is no loss of generality in assuming that $\mathcal{N} = V^{-1}([0, c_0])$ where V is taken from Lemma 4 and $c_0 \in (0, 1)$. By a simple compactness argument, the chain of inclusions

$$\Phi(\tau, \mathcal{N}) \subset V^{-1}([0, c/2]) \subset V^{-1}([0, c]) \subset B(\mathcal{A}, \varepsilon)$$

holds true for some $\tau = \tau(\varepsilon) > 1$ and $c = c(\varepsilon) \in (0, c_0]$. Assuming $\eta < 1$, there exists a (uniquely defined) $M \in \mathbb{N}$ such that $t_M < \tau \leq t_{M+1} < \tau + 1$. In view of inequality (4), it follows that

$$|\varphi(h_{M+1}, \dots, h_1; \cdot) - \Phi(h_{M+1} + \dots + h_1; \cdot)| < K_0(\tau + 1) \cdot \eta^p.$$

In particular,

$$\mathcal{N}_{t_{M+1}} \subset B(V^{-1}([0, c/2]), K_0(\tau + 1) \cdot \eta^p) \subset V^{-1}([0, c])$$

for η sufficiently small, say $\eta < \kappa$ with some positive $\kappa = \kappa(c)$. Requiring that $\eta < \min\{\kappa, h^*(c)\}$, the last assertion of Lemma 4 implies that $\mathcal{N}_{t_m} \subset B(\mathcal{A}, \varepsilon)$ whenever $m > M$. QED

Remark 2. Example 1 shows that the sequence of sets $\{\mathcal{N}_{t_m}\}_{m=1}^\infty$ need not converge in the Hausdorff metric. However, replacing $\mathcal{N}_{t_m} = \varphi(h_m, \dots, h_1; \mathcal{N})$ by $\mathcal{N}^{t_m} = \varphi(h_1, \dots, h_m; \mathcal{N})$ (i.e. reversing the order of the individual timesteps), the new sequence of sets $\{\mathcal{N}^{t_m}\}_{m=1}^\infty$ converges whenever $\sum_m h_m = \infty$. The result is a trivial consequence of Lemmata 2 and 3 if $\mathcal{N} = V^{-1}([0, c_0])$ for some $c_0 \in (0, 1)$. In fact, $\{\mathcal{N}^{t_m}\}_{m=1}^\infty$ forms a nested sequence of compacta in $\mathcal{N} = V^{-1}([0, c_0])$ and, a fortiori, $\limsup_m \mathcal{N}^{t_m} = \liminf_m \mathcal{N}^{t_m} = \bigcap_m \mathcal{N}^{t_m}$. In the general case, however, $\{\mathcal{N}^{t_m}\}_{m=1}^\infty$ is not a nested sequence of compacta. Nevertheless, $\limsup_m \mathcal{N}^{t_m} = \liminf_m \mathcal{N}^{t_m}$ is still true. The proof is an easy application of Theorem 1. In fact, let \mathcal{N} be a compact neighborhood of \mathcal{A} in $R(\mathcal{A})$ and choose $c_0 \in (0, 1)$ in such a way that $V^{-1}([0, c_0]) \subset \mathcal{N}$. Given an integer $m \geq 1$ arbitrarily, we conclude there exists a $K = K(c_0, \mathcal{N}, m) \in \mathbb{N}$ such that

$$\varphi(h_1, \dots, h_{m+k}; \mathcal{N}) \subset \varphi(h_1, \dots, h_m; V^{-1}([0, c_0])) \subset \varphi(h_1, \dots, h_m; \mathcal{N}) \tag{7}$$

whenever $k = K, K + 1, \dots$. But the double inclusion (7) implies that

$$\limsup_m \mathcal{N}^{t_m} = \limsup_m \varphi(h_1, \dots, h_m; V^{-1}([0, c_0]))$$

and similarly, also the limes inferiors are equal. The convergence of $\{\mathcal{N}^{t_m}\}_{m=1}^\infty$ in the Hausdorff metric is called pullback convergence and, under the additional condition $\sup\{h_k/h_\ell \mid k, \ell = 1, 2, \dots\} < \infty$, has been established by Kloeden and Schmalfuss [17] for the first time. (In an interesting sequence of papers, Kloeden and his coworkers [16], [17], [18], [6], [15] discuss a more general concept, the concept of a cocycle attractor. Most of their results are stated and proved for abstract cocycles with the shift operator acting on a compact parameter space. Inequality $\sup\{h_k/h_\ell \mid k, \ell = 1, 2, \dots\} < \infty$ (or, equivalently, inequality $\liminf_m h_m > 0$) is a compactness assumption on the parameter space $\{(h_1, h_2, \dots)\}$ with the shift operator $\theta : (h_1, h_2, \dots) \rightarrow (h_2, h_3, \dots)$ but, as it is demonstrated above, plays no role at all and can be omitted. (This is implicit in proving the main results in [17] and [15] but does not seem to be explicitly mentioned in the literature prior to the paper [10]. The use of $\limsup_m \mathcal{N}^{t_m}$ and $\liminf_m \mathcal{N}^{t_m}$ above is new and seems to simplify several arguments in the general theory of cocycle attractors, too.))

Theorem 2. Assume that \mathcal{A} is an attractor for (1) and let \mathcal{N} be a compact neighborhood of \mathcal{A} in $R(\mathcal{A})$. Given a stepsize sequence h_1, h_2, \dots with the properties that $h_m \rightarrow 0$ and $\sum_m h_m = \infty$, then $\mathcal{N}_{t_m} \rightarrow \mathcal{A}$ in an upper semicontinuous way.

Proof. Since \mathcal{A} is the maximal Φ -invariant set in \mathcal{N} , property (iv)_U implies it is sufficient to prove that $\limsup_m \mathcal{N}_{t_m}$ is Φ -invariant. In other words, it remains to point out that $\Phi(T, x) \in \limsup_m \mathcal{N}_{t_m}$ whenever $T \in \mathbb{R}$ and $x \in \limsup_m \mathcal{N}_{t_m}$. (Essentially the same argument shows that $\liminf_m \mathcal{N}_{t_m}$ is Φ -invariant, too.)

We distinguish two cases according as $T > 0$ or $T < 0$. (Case $T = 0$ is trivial.)

Consider first the case $T > 0$. Given $\varepsilon > 0$ and $\eta \in (0, T)$ arbitrarily, property (6) implies there exists a $q \in \mathcal{N}$ and an integer $N \geq 1$ such that

$$|\varphi(h_N, \dots, h_1; q) - x| < \varepsilon \quad \text{and} \quad \sup\{h_{N+1}, h_{N+2}, \dots\} < \eta.$$

Choose integer $M > N$ in such a way that $T - \eta < h_{N+1} + \dots + h_M < T$. By using properties (4), (3), (2), it follows that

$$\begin{aligned} & |\varphi(h_M, \dots, h_{N+1}, h_N, \dots, h_1; q) - \Phi(T, x)| \leq \\ & |\varphi(h_M, \dots, h_{N+1}; \varphi(h_N, \dots, h_1; q)) - \Phi(h_M + \dots + h_{N+1}, \varphi(h_N, \dots, h_1; q))| \\ & + |\Phi(h_M + \dots + h_{N+1}, \varphi(h_N, \dots, h_1; q)) - \Phi(h_M + \dots + h_{N+1}, x)| \\ & + |\Phi(h_M + \dots + h_{N+1}, x) - \Phi(T, x)| \leq K_0(T) \cdot \eta^p + e^{\Gamma T} \varepsilon + \gamma \eta. \end{aligned}$$

In view of property (6), $\Phi(T, x) \in \limsup_m \mathcal{N}_{t_m}$ is immediate.

Consider now the second case $T = -\tau < 0$. Given $\varepsilon > 0$ and $\eta \in (0, \tau)$ arbitrarily, property (6) implies that, for some $q \in \mathcal{N}$ and integers $N > M \geq 1$ suitably chosen, $\tau - \eta < h_{M+1} + \dots + h_N < \tau$ as well as

$$|\varphi(h_N, \dots, h_1; q) - x| < \varepsilon \quad \text{and} \quad \sup\{h_{M+1}, h_{M+2}, \dots\} < \eta.$$

By a consecutive application of (3), (2) and (4), we obtain that

$$\begin{aligned} & |\varphi(h_M, \dots, h_1; q) - \Phi(-\tau, x)| \\ & \leq e^{\Gamma \tau} \cdot |\Phi(\tau, \varphi(h_M, \dots, h_1; q)) - \Phi(\tau, \Phi(-\tau, x))| \\ & \leq e^{\Gamma \tau} \cdot (|\Phi(\tau, \varphi(h_M, \dots, h_1; q)) - \Phi(h_N + \dots + h_{M+1}, \varphi(h_M, \dots, h_1; q))| \\ & + |\Phi(h_N + \dots + h_{M+1}, \varphi(h_M, \dots, h_1; q)) - \varphi(h_N, \dots, h_{M+1}; \varphi(h_M, \dots, h_1; q))| \\ & + |\varphi(h_N, \dots, h_1; q) - x|) \leq e^{\Gamma \tau} (\gamma \eta + K_0(\tau) \cdot \eta^p + \varepsilon). \end{aligned}$$

Hence $\Phi(-\tau, x) = \Phi(T, x) \in \limsup_m \mathcal{N}_{t_m}$ by property (6). QED

Theorem 3. *Assume that \mathcal{A} is an attractor for (1) and let \mathcal{N} be a compact neighborhood of \mathcal{A} in $R(\mathcal{A})$. Then there exists a stepsize sequence h_1, h_2, \dots with the properties that $h_m \rightarrow 0$, $\sum_m h_m = \infty$, and $\mathcal{N}_{t_m} \supset \mathcal{A}$ for $m = 1, 2, \dots$. (In view of Theorem 2, this implies immediately that $d_H(\mathcal{N}_{t_m}, \mathcal{A}) \rightarrow 0$.)*

Proof. As in the proof of Theorem 1, we may assume that $\mathcal{N} = V^{-1}([0, c_0])$ where V is taken from Lemma 4 and $c_0 \in (0, 1)$.

With positive integers $N(0), N(1), \dots$ specified later, we set

$$M(0) = 0 \quad \text{and} \quad M(k+1) = M(k) + N(k) \quad \text{for each } k \in \mathbb{N}.$$

Every positive integer can be uniquely represented in as

$$m = M(k(m)) + \ell(m) \quad \text{with some } k(m) \in \mathbb{N}, \ell(m) \in \{1, 2, \dots, N(k(m))\}.$$

For $m = 1, 2, \dots$ define $h_m = 1/N(k(m))$ and note that $j = k(m)$ if and only if $m = M(j+1), \dots, M(j+1)$, $j \in \mathbb{N}$.

It is sufficient to prove that, with $\{N(k)\}_{k=0}^\infty$ suitably chosen, inclusion

$$\Phi(t_m + \sum_{i=0}^{k(m)} 2^{-i}, \mathcal{N}) \subset \mathcal{N}_{t_m} \tag{8}$$

holds true for each m . We proceed by induction. For simplicity, we set $t_0 = 0$, $k(0) = -1$ and (recalling $\sum_{i=0}^{-1} 2^{-i} = 0$) observe that (8) is satisfied for $m = 0$. Induction on $m = M(k) + \ell$ can be replaced by a double induction on $k \in \mathbb{N}$ and

$\ell \in \{1, 2, \dots, N(k)\}$. Thus the induction hypothesis is that (8) is true for some $m = M(k)$ or, equivalently,

$$\Phi(k + \sum_{i=0}^{k-1} 2^{-i}, \mathcal{N}) \subset \mathcal{N}_{t_{M(k)}} \quad \text{for some } k \in \mathbb{N}. \tag{9}$$

In view of inequality (4), we have that

$$\left| \varphi^\ell \left(\frac{1}{N(k)}, x \right) - \Phi \left(\frac{\ell}{N(k)}, x \right) \right| \leq K_0(1) \cdot \left(\frac{1}{N(k)} \right)^p \tag{10}$$

whenever $x \in \mathbb{R}^n$ and $\ell = 1, 2, \dots, N(k)$.

Let \mathcal{S} and $\tilde{\mathcal{S}}$ be compact subsets of \mathbb{R}^n and assume that $\tilde{\mathcal{S}}$ is a compact neighborhood of \mathcal{S} . Since Φ is continuous and $\Phi(t, \cdot)$ is a self-diffeomorphism of \mathbb{R}^n , a standard compactness argument implies that $\{(t, \Phi(t, \tilde{\mathcal{S}})) \mid t \in [0, 1]\}$ is a compact neighborhood of the compact set $\{(t, \Phi(t, \mathcal{S})) \mid t \in [0, 1]\}$ in $[0, 1] \times \mathbb{R}^n$.

With $\mathcal{S} = \Phi(k + \sum_{i=0}^k 2^{-i}, \mathcal{N})$ and $\tilde{\mathcal{S}} = \Phi(k + \sum_{i=0}^{k-1} 2^{-i}, \mathcal{N})$, it follows via inequality (10) that for $N(k)$ sufficiently large

$$\varphi^\ell \left(\frac{1}{N(k)}, \partial\Phi(k + \sum_{i=0}^{k-1} 2^{-i}, \mathcal{N}) \right) \subset \Phi \left(\frac{\ell}{N(k)}, \mathbb{R}^n \setminus \Phi(k + \sum_{i=0}^k 2^{-i}, \mathcal{N}) \right)$$

and, a fortiori, since $\varphi^\ell(1/N(k), \cdot)$ is a self-diffeomorphism of \mathbb{R}^n ,

$$\begin{aligned} &\Phi \left(\frac{\ell}{N(k)} + k + \sum_{i=0}^k 2^{-i}, \mathcal{N} \right) = \Phi \left(\frac{\ell}{N(k)}, \Phi(k + \sum_{i=0}^k 2^{-i}, \mathcal{N}) \right) \\ &\subset \varphi^\ell \left(\frac{1}{N(k)}, \Phi(k + \sum_{i=0}^{k-1} 2^{-i}, \mathcal{N}) \right) \quad \text{whenever } \ell = 1, 2, \dots, N(k). \end{aligned}$$

By the induction hypothesis, we conclude that

$$\Phi \left(\frac{\ell}{N(k)} + k + \sum_{i=0}^k 2^{-i}, \mathcal{N} \right) \subset \varphi^\ell \left(\frac{1}{N(k)}, \mathcal{N}_{t_{M(k)}} \right) \tag{11}$$

whenever $\ell = 1, 2, \dots, N(k)$. By the construction, inclusion (11) is identical with case $m = M(k) + \ell$ of (8), $\ell = 1, 2, \dots, N(k)$. In particular, case $\ell = N(k)$ of (11) replaces k by $k + 1$ in the induction hypothesis (9) and we are done. QED

Remark 3. Let h_1, h_2, \dots be the stepsize sequence constructed in the proof of Theorem 3. Then $d_H(\mathcal{N}'_{t'_m}, \mathcal{A}) \rightarrow 0$ holds true for all stepsize sequences h'_1, h'_2, \dots satisfying $\sum_m h'_m = \infty, 0 \leq h'_m \leq h_m, m = 1, 2, \dots$

Starting from a compact set $\mathcal{N} \subset \mathbb{R}^n$ satisfying $\Phi(t, \mathcal{N}) \subset \text{int}(\mathcal{N})$ for each $t > 0$, the variable stepsize discretization method in Theorem 3 is an approximation procedure that can be used for establishing attractor $\mathcal{A} = \cap\{\Phi(t, \mathcal{N}) \mid t \geq 0\}$. Exploiting the concept of attractor-repellor pairs, the simple time-reversal trick shows that the very same variable stepsize argument leads to an approximation procedure that can be used for establishing $R(\mathcal{A})$. The general problem of finding the entire region of attraction $R(\mathcal{A})$ is extremely difficult and has a long history. For details, see the references in Grüne [13]. See also [7]. We are indebted to Professor S. Maruster for pointing out this latter reference.

Corollary 2. *Assume that \mathcal{A} is an attractor for (1) and let \mathcal{N} be a compact neighborhood of \mathcal{A} in $R(\mathcal{A})$. Reversing time, we consider also the differential equation $\dot{x} = -f(x)$, its solution flow Ψ (defined by letting $\Psi(t, x) = \Phi(-t, x)$ whenever $(t, x) \in \mathbb{R} \times \mathbb{R}^n$) and a corresponding C^{p+k+1} discretization operator ψ (which can be defined e.g. by letting $\psi(h, \cdot) = \varphi^{-1}(h, \cdot)$ whenever $h \in [0, h_0]$). Then there exists a stepsize sequence h_1, h_2, \dots with the properties that $h_m \rightarrow 0$, $\sum_m h_m = \infty$ and $\cup_m \psi(h_m, \dots, h_1; \mathcal{N}) = R(\mathcal{A})$.*

Proof. By letting $\hat{\Psi}(t, \{\infty\}) = \{\infty\}$ for each $t \in \mathbb{R}$ (resp. $\hat{\psi}(h, \{\infty\}) = \{\infty\}$ for each $h \in [0, h_0]$), the dynamical system Ψ (resp. the discretization operator ψ) extends to $\mathbb{R}^n \cup \{\infty\}$, the one-point compactification of \mathbb{R}^n . It is well-known that $\hat{\mathcal{R}} = (\mathbb{R}^n \cup \{\infty\}) \setminus R(\mathcal{A})$ is an attractor for $\hat{\Psi}$ whose region of attraction is $(\mathbb{R}^n \cup \{\infty\}) \setminus \mathcal{A}$. Since $\hat{\psi}(h_m, \dots, h_1; \mathcal{N})$ is a self-homeomorphism of $\mathbb{R}^n \cup \{\infty\}$ (for any finite sequence h_1, h_2, \dots in $[0, h_0]$), the desired result follows via Theorem 3. QED

All the previous results are demonstrated by the following Example which goes back to [3], one of the first computer experiment on numerical bifurcations. The reason for giving some details below is a general lack of rigorous examples in the literature.

Example 2. With a C^∞ function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ specified later and $\rho^2 = x^2 + y^2$, consider the planar differential equation

$$\left. \begin{aligned} \dot{x} &= y + xf(\rho) \\ \dot{y} &= -x + yf(\rho) \end{aligned} \right\} \iff \text{ (in polar coordinates) } \begin{cases} \dot{\rho} = \rho f(\rho) \\ \dot{\theta} = -1 \end{cases} \quad (12)$$

and apply the explicit Euler method. The induced discretization operator is of the form

$$\varphi_E \left(h, \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} x \\ y \end{pmatrix} + h \begin{pmatrix} y + xf(\rho) \\ -x + yf(\rho) \end{pmatrix} \quad \text{whenever} \quad \left(h, \begin{pmatrix} x \\ y \end{pmatrix} \right) \in [0, h_0] \times \mathbb{R}^2.$$

Note that the origin $\mathbf{0} = \text{col}(0, 0)$ is an equilibrium for (12) and remains an equilibrium for $\varphi_E(h, \cdot)$. Clearly $\varphi_E(h, \cdot)$ preserves the rotational symmetry and

$$\left| \begin{pmatrix} x \\ y \end{pmatrix} \right| = \rho \quad \text{implies that} \quad \left| \varphi_E \left(h, \begin{pmatrix} x \\ y \end{pmatrix} \right) \right|^2 = \rho^2 ((1 + hf(\rho))^2 + h^2). \quad (13)$$

In particular, assuming $h_0 < 1$, we see that the circle $S_r = \{(\rho, \theta) \mid \rho = r\}$ of radius $r > 0$ is $\varphi_E(h, \cdot)$ -invariant if and only if

$$2f(r) + h(1 + f^2(r)) = 0 \quad \text{i.e.} \quad f(r) = h^{-1} \cdot (-1 \pm (1 - h^2)^{1/2}). \quad (14)$$

To exclude eventual large roots, we assume for simplicity that $f(\rho) > -1$ for each $\rho \in \mathbb{R}^+$. Thus the "−" case within the "±"-sign of (14) is irrelevant and

$$f(r) = \sum_{k=1}^{\infty} (-1)^k \binom{1/2}{k} h^{2k-1} = -\frac{h}{2} - \frac{h^3}{8} + \mathcal{O}(h^5) \quad \text{for } h \in (0, h_0].$$

From now on, assume that $f(\rho) = (1 - \rho)^2$ whenever $\rho \in [\frac{1}{2}, \frac{3}{2}]$, $f(\rho) = 2 - \rho$ whenever $\rho \in [\frac{7}{4}, \frac{5}{2}]$, $f^{-1}(0) = \{1, 2\}$, and that f is convex and decreasing on $[2, \infty)$. Clearly the annulus $\mathcal{A} = \{(\rho, \theta) \mid 1 \leq \rho \leq 2\}$ is an attractor for (12), $R(\mathcal{A}) = \mathbb{R}^2 \setminus \{\mathbf{0}\}$, $\partial\mathcal{A} = S_1 \cup S_2$, both S_1 and S_2 are periodic orbits, and S_2 is an attractor with $R(S_2) = \mathbb{R}^2 \setminus \{(\rho, \theta) \mid \rho \leq 1\}$.

In what follows we present several observations on explicit Euler discretizations of (12) with various stepsize sequences. For simplicity, we set $\mathcal{N} = \{(\rho, \theta) \mid \frac{1}{2} \leq \rho \leq \frac{5}{2}\}$ and assume that h_0 is small enough to imply that $\varphi_E(h, \cdot)$ is a self-diffeomorphism of \mathbb{R}^2 , $\varphi_N(h, \mathcal{N}) \subset \text{int}(\mathcal{N})$, and that equation (14) has a unique root $r = r_h$ in $[2, \frac{5}{2}]$ whenever $h \in (0, h_0]$. (By the construction, $r_h = 2 + h^{-1} \cdot (-1 + (1 - h^2)^{1/2})$.)

A.) We consider first the constant stepsize case and demonstrate the Kloeden–Lorenz Theorem recalled in the second paragraph of this Section. For $h \in (0, h_0]$, it is readily seen that $\mathcal{A}_h = S_{r_h}$ and $R(S_{r_h}) = \mathbb{R}^2 \setminus \{\mathbf{0}\}$. Geometrically, it means that annulus \mathcal{A} (as an attractor for (12)) collapses to the circle S_{r_h} (as an attractor for $\varphi_E(h, \cdot)$) and $d_H(S_{r_h}, S_2) \rightarrow 0$ as $h \rightarrow 0$: the convergence $S_{r_h} \rightarrow \mathcal{A}$ is upper but not lower semicontinuous.

Remark 4. The situation is completely different for $\varphi_I(h, \cdot)$, the discrete-time dynamical system coming from the implicit Euler method. Then \mathcal{A}_h is an annulus and $d_H(\mathcal{A}_h, \mathcal{A}) \rightarrow 0$ as $h \rightarrow 0$. It is worth mentioning that \mathcal{A}_h can be computed explicitly. The reason is that a circle S_r , $r > 0$ is $\varphi_I(h, \cdot)$ -invariant if and only if $-2f(r) + h(1 + f^2(r)) = 0$.

B.) Consider a variable stepsize sequence h_1, h_2, \dots and assume that $h_m \rightarrow 0$, $\sum_m h_m = \infty$. Obviously,

$$\mathcal{N}_{t_m} = \{(\rho, \theta) \mid \rho_m \leq \rho \leq \tilde{\rho}_m\}$$

where the sequences $\{\rho_m\}_m$ and $\{\tilde{\rho}_m\}_m$ are defined by the respective recursions (cf. (13))

$$\rho_1 = 1/2 \quad \text{and} \quad \rho_{m+1} = \rho_m((1 + 2h_m f(\rho_m))^2 + h_m^2)^{1/2}, \quad m = 1, 2, \dots$$

and

$$\tilde{\rho}_1 = 5/2 \quad \text{and} \quad \tilde{\rho}_{m+1} = \tilde{\rho}_m((1 + 2h_m f(\tilde{\rho}_m))^2 + h_m^2)^{1/2}, \quad m = 1, 2, \dots$$

It is readily checked that $\tilde{\rho}_m > 2$ for each m and $\tilde{\rho}_m \rightarrow 2$ as $m \rightarrow \infty$ (but the convergence is not necessarily decreasing). Similarly, we obtain that

$$\lim_{m \rightarrow \infty} \rho_m = \begin{cases} 2 & \text{if } \rho_m > 1 \text{ for some } m \\ 1 & \text{if } \rho_m \leq 1 \text{ for all } m \end{cases}$$

(but only the convergence to limit 1 is necessarily increasing). Thus

$$d_H(\mathcal{N}_{t_m}, S_2) \rightarrow 0 \quad \text{or} \quad d_H(\mathcal{N}_{t_m}, \mathcal{A}) \rightarrow 0 \quad \text{as } m \rightarrow \infty. \tag{15}$$

Note that the only nontrivial Φ -invariant compact sets in \mathcal{N} with rotational symmetry are \mathcal{A} , S_1 , S_2 , and $S_1 \cup S_2$. Thus (15) is in full agreement with the Φ -invariance of $\limsup_m \mathcal{N}_{t_m}$ and $\liminf_m \mathcal{N}_{t_m}$ observed in proving Theorem 2 and with parts $(iv)_U$ and $(iv)_L$ of the Lemmata in Subsection 1.2. An alternative way of deriving (15) is to refer to these abstract results above and to check solely the simple inequality $\tilde{\rho}_m > 2$, for each m .

B.1.) Consider the variable stepsize sequence $h_m = c/m^{1/2}$, $m = 1, 2, \dots$ where $c \in (0, h_0]$ is a parameter. Then $d_H(\mathcal{N}_{t_m}, S_2) \rightarrow 0$. In fact, as long as $0 < \rho_m \leq 2$, we have that

$$\rho_{m+1}^2 \geq \rho_1^2 \prod_{j=1}^m (1 + h_j^2) = \rho_1^2 \prod_{j=1}^m \left(1 + \frac{c^2}{j}\right)$$

and, a fortiori, $\rho_m > 1$ for some m . Thus the results in Part B.) apply.

B.2.) Consider the variable stepsize sequence $h_m = c/m$, $m = 1, 2, \dots$ where $c \in (0, \min\{h_0, 1/4\}]$ is a parameter. Then $d_H(\mathcal{N}_{t_m}, \mathcal{A}) \rightarrow 0$. Referring to the results in Part B.), we see it is enough to prove that $\rho_m \leq 1$ for each m .

In fact, as long as $\frac{1}{2} \leq \rho_m \leq \frac{3}{2}$, we have that

$$\rho_{m+1} = \rho_m \left(\left(1 + \frac{c}{m}(1 - \rho_m)^2\right)^2 + \frac{c^2}{m^2} \right)^{1/2} \leq \rho_m + \frac{c}{m}(1 - \rho_m) + \frac{c^2}{m^2}.$$

Consider now the linear recursion

$$\sigma_1 = 1/2 \quad \text{and} \quad \sigma_{m+1} = \left(1 - \frac{c}{m}\right) \sigma_m + \frac{c}{m} + \frac{c^2}{m^2}, \quad m = 1, 2, \dots$$

As long as $\frac{1}{2} \leq \rho_m \leq \frac{3}{2}$, a simple induction shows that $\rho_m \leq \sigma_m$. But, again by induction on m , $\sigma_m \leq 1 - \frac{1}{2m}$ for each m . This is certainly true for the starting case $m = 1$. The induction hypothesis leads to checking inequality

$$\left(1 - \frac{c}{m}\right) \left(1 - \frac{1}{2m}\right) + \frac{c}{m} + \frac{c^2}{m^2} \leq 1 - \frac{1}{2(m+1)}$$

or, equivalently, $c + 2c^2 \leq m(1 - c - 2c^2)$ for $m = 1, 2, \dots$ which is clearly satisfied if $c \leq 1/4$. Hence $\rho_m \leq \sigma_m < 1$ for all m .

C.) Consider now a variable stepsize sequence h_1, h_2, \dots and assume that $\rho_m > 1$ for some m . Assuming $h_0 < 1/4$, we can choose indices $1 < M < N$ in such a way that

$$\rho_{M-1} < 1 - h_0^{1/2} \leq \rho_M < \rho_{M+1} < \dots < \rho_N \leq 1 < \rho_{N+1}.$$

Observe that

$$\rho_{m+1}^2 \leq \rho_m^2((1 + h_k h_0)^2 + h_k^2) \leq 1 + 4h_k h_0 \quad \text{whenever } m = M, M+1, \dots, N$$

and, similarly, $\rho_M^2 \leq \rho_{M-1}^2 \leq 1 + 2h_0$. It follows that

$$\left(\frac{1}{1 - h_0^{1/2}}\right)^2 \leq \frac{\rho_{N+1}^2}{\rho_{M-1}^2} \leq (1 + 2h_0) \prod_{j=M}^N (1 + 4h_j h_0).$$

Taking the logarithm of both sides, we conclude via the simple inequality $1 + x \leq e^x \leq (1 - x)^{-1}$ (valid for $0 \leq x \leq 1$) that

$$1 + 2 \sum_{j=M}^N h_k \geq \frac{1}{h_0^{1/2}},$$

an inequality to be analyzed in the next subsection.

2.2. A new characterization of lower semicontinuity. Parts A.) and B.1) – B.2) of Example 2 show that, depending on how the stepsize sequence is chosen, attractors may collapse under discretization. However — and this is the meaning of the last inequality in Part C.) — in case the attractor actually collapses, the smaller the stepsize, the larger is the total time $\sum_{j=1}^N h_j \geq \sum_{j=M}^N h_j$ the discretization process needs to make the attractor collapsed. Moreover, with the maximal stepsize approaching zero, this total time goes to infinity. In this sense, all attractors are robust. Even if they collapse under discretization, the time behaviour of the discretization procedure shows where they collapsed from. Our next result formulates this latter observation in a precise manner.

Theorem 4. Assume that \mathcal{A} is an attractor for (1) and let \mathcal{N} be a compact neighborhood of \mathcal{A} in $R(\mathcal{A})$. Then the following statements are pairwise equivalent.

- (i): For every $\varepsilon > 0$ there exist an $H_1(\varepsilon) > 0$ and a time $\sigma_1(\varepsilon) > 0$ with the property that, given a stepsize sequence h_1, h_2, \dots with $h_0 < H_1(\varepsilon)$, then $\mathcal{A} \subset B(\mathcal{N}_{t_m}, \varepsilon)$ whenever $t_m \geq \sigma_1(\varepsilon)$.
- (ii): For every $\varepsilon > 0$ there exist an $H_2(\varepsilon) > 0$ and a time $\sigma_2(\varepsilon) > 0$ with the property that, given two stepsize sequences h_1, h_2, \dots and h'_1, h'_2, \dots with $h_0, h'_0 < H_2(\varepsilon)$, then $\mathcal{N}_{t'_m} \subset B(\mathcal{N}_{t_m}, \varepsilon)$ whenever $t_m, t'_m \geq \sigma_2(\varepsilon)$.
- (iii): For every $\varepsilon > 0$ there exist an $H_3(\varepsilon) > 0$ and a time $\sigma_3(\varepsilon) > 0$ with the property that, given a stepsize sequence h_1, h_2, \dots with $h_0 < H_3(\varepsilon)$, then $\mathcal{N}_{t'_m} \subset B(\mathcal{N}_{t_m}, \varepsilon)$ whenever $t_m, t'_m \geq \sigma_3(\varepsilon)$.

Proof. There is no loss of generality in assuming that $\mathcal{N} = V^{-1}([0, c_0])$ where V is taken from Lemma 4 and $c_0 \in (0, 1)$.

(i) \Rightarrow (ii) : Let $\vartheta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a strictly decreasing continuous function with the properties that $\lim_{t \rightarrow \infty} \vartheta(t) = 0$ and

$$\mathcal{A} \subset \Phi(t, \mathcal{N}) \subset B(\mathcal{A}, \vartheta(t)) \quad \text{whenever } t \geq 0.$$

The existence of ϑ follows from a standard compactness argument. Given $\varepsilon > 0$, fix $T(\varepsilon) > 0$ in such a way that $\vartheta(\frac{T(\varepsilon)}{2}) = \varepsilon$.

As a simple consequence of inequality (4), there exists a $\chi(\varepsilon) > 0$ such that

$$\varphi(h'_{k'}, \dots, h'_1; \mathcal{N}) \subset B(\Phi(T(\varepsilon), \mathcal{N}), K_0(T(\varepsilon) + 1) \cdot \chi^p(\varepsilon)) \subset \Phi(\frac{T(\varepsilon)}{2}, \mathcal{N})$$

whenever $h'_0 \leq \chi(\varepsilon)$ and $t'_{k'-1} < T(\varepsilon) \leq t'_{k'} < T(\varepsilon) + 1$. Applying Corollary 1 (and passing to a smaller $\chi(\varepsilon)$ if necessary), it follows that

$$\mathcal{N}'_{t'_m} = \varphi(h'_{m'}, \dots, h'_{k'}, \dots, h'_1; \mathcal{N}) \subset \Phi(\frac{T(\varepsilon)}{2}, \mathcal{N}) \subset B(\mathcal{A}, \vartheta(\frac{T(\varepsilon)}{2}))$$

whenever $m' \geq k'$ or, equivalently, $t'_{m'} \geq T(\varepsilon)$.

In view of property (i), we conclude that

$$\mathcal{N}'_{t'_m} \subset B(\mathcal{A}, \varepsilon) \subset B(B(\mathcal{N}_{t_m}, \varepsilon), \varepsilon) = B(\mathcal{N}_{t_m}, 2\varepsilon)$$

whenever $h_0 \leq H_1(\varepsilon)$ and $t_m \geq \sigma_1(\varepsilon)$. Thus $H_2(\varepsilon)$ and $\sigma_2(\varepsilon)$ can be chosen for $\min\{\chi(\frac{\varepsilon}{2}), H_1(\frac{\varepsilon}{2})\}$ and $\max\{T(\frac{\varepsilon}{2}), \sigma_1(\frac{\varepsilon}{2})\}$, respectively.

(ii) \Rightarrow (iii) : Observe that (iii) is just the $h'_1 = h_1, h'_2 = h_2, \dots$ special case of (ii).

(iii) \Rightarrow (i) : As a simple consequence of inequality (4),

$$\mathcal{A} \subset \Phi(\sigma_3(\varepsilon) + 1, \mathcal{N}) \subset B(\varphi(h_{m'}, \dots, h_1; \mathcal{N}), K_0(\sigma_3(\varepsilon) + 1) \cdot H_3^p(\varepsilon))$$

whenever $h_0 \leq H_3(\varepsilon)$ and $t_{m'-1} < \sigma_3(\varepsilon) \leq t_{m'} < \sigma_3(\varepsilon) + 1$. By passing to a smaller $H_3(\varepsilon)$ if necessary, we may assume that $K_0(\sigma_3(\varepsilon) + 1) \cdot H_3^p(\varepsilon) \leq \varepsilon$. In view of property (iii), it follows that

$$\mathcal{A} \subset B(\mathcal{N}_{t'_m}, \varepsilon) \subset B(B(\mathcal{N}_{t_m}, \varepsilon), \varepsilon) = B(\mathcal{N}_{t_m}, 2\varepsilon)$$

whenever $t_m \geq \sigma_3(\varepsilon)$. QED

Though the following result is not a formal consequence of Theorem 4, we feel justified to call it as

Corollary 3. Assume that \mathcal{A} is an attractor for (1) and let \mathcal{N} be a compact neighborhood of \mathcal{A} in $R(\mathcal{A})$. Then the following statements are pairwise equivalent.

- (a): $\mathcal{A}_h \rightarrow \mathcal{A}$ in a lower semicontinuous way.
 (b): For every $\varepsilon > 0$ there exist an $H(\varepsilon) > 0$ and a time $T(\varepsilon) > 0$ such that $\mathcal{A} \subset B(\varphi^m(h, \mathcal{N}), \varepsilon)$ whenever $h \leq H(\varepsilon)$ and $mh \geq T(\varepsilon)$.
 (c): For every $\varepsilon > 0$ there exist an $H(\varepsilon) > 0$ and a time $T(\varepsilon) > 0$ such that

$$\varphi^{m'}(h', \mathcal{N}) \subset B(\varphi^m(h, \mathcal{N}), \varepsilon) \quad (16)$$

whenever $h, h' \leq H(\varepsilon)$ and $mh, m'h' \geq T(\varepsilon)$.

- (d): For every $\varepsilon > 0$ there exist an $H(\varepsilon) > 0$ and a time $T(\varepsilon) > 0$ such that $\varphi^m(h, \mathcal{N}) \subset B(\mathcal{A}_h, \varepsilon)$ whenever $h \leq H(\varepsilon)$ and $mh \geq T(\varepsilon)$.

Proof. When repeated for constant stepsizes, the proof of Theorem 4 shows that properties (b), (c), and

- (\tilde{d}): for every $\varepsilon > 0$ there exist an $H(\varepsilon) > 0$ and a time $T(\varepsilon) > 0$ such that $\varphi^{m'}(h, \mathcal{N}) \subset B(\varphi^m(h, \mathcal{N}), \varepsilon)$ whenever $h \leq H(\varepsilon)$ and $mh, m'h \geq T(\varepsilon)$

are pairwise equivalent. The equivalence of (\tilde{d}) and (d) follows via $\mathcal{A}_h = \cap \{\varphi^m(h, \mathcal{N}) \mid m \in \mathcal{N}\}$. On the other hand, by letting $m \rightarrow \infty$, we see that (a) is implied by (b). Finally, (b) is a consequence of (a) when combined with $\mathcal{A}_h \subset \varphi^m(h, \mathcal{N}) \subset B(\mathcal{A}, \varepsilon)$ which is just the constant stepsize special case of Theorem 1. QED

Since \mathcal{A} and \mathcal{A}_h are usually unknown, the statement among (a), (b), (c), and (d) with the highest practical value in computer experiments is (c). Note that the equivalence between (a) and (d) is identical with result (\mathcal{C}^*) on uniform attraction rates recalled in the second paragraph of the Introduction. Part C.) of Example 2 shows how nonuniform attraction rates appear in practice. The equivalence between (a) and (c) is new.

Remark 5. With inclusion (16) replaced by inclusion $\mathcal{X} \subset B(\mathcal{Y}, \varepsilon)$ where

$$\mathcal{X}, \mathcal{Y} \in \{\mathcal{A}, \mathcal{A}_h, \mathcal{A}_{h'}, \varphi^m(h, \mathcal{N}), \varphi^{m'}(h', \mathcal{N}), \Phi^m(h, \mathcal{N}), \Phi^{m'}(h', \mathcal{N})\},$$

it is a nice exercise to investigate if the respective (\mathcal{X}, \mathcal{Y})-version of (c) is equivalent to lower semicontinuity. Including the symmetric cases, there are altogether $7^2 = 49$ possibilities. It is not hard to check that 16 of them are equivalent to lower semicontinuity. Out of the remaining 33 cases 15 cases (like $\mathcal{A} \subset B(\mathcal{A}, \varepsilon)$, $\mathcal{A}_h \subset B(\varphi^m(h, \mathcal{N}), \varepsilon)$, or $\Phi^m(h, \mathcal{N}) \subset B(\Phi^{m'}(h', \mathcal{N}), \varepsilon)$) are trivial, 12 cases are equivalent to upper semicontinuity, and all the remaining 6 cases (like $\mathcal{A}_h \subset B(\mathcal{A}_{h'}, \varepsilon)$, or $\mathcal{A}_{h'} \subset B(\varphi^m(h, \mathcal{N}), \varepsilon)$) are equivalent to statement

$$\lim_{h \rightarrow 0} d_H(\mathcal{A}_h, \mathcal{A}^*) = 0 \quad \text{for some } \mathcal{A}^* \in \mathcal{C}_H(\mathcal{A}).$$

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