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Energy Decay for the Strongly Damped Nonlinear Beam Equation and Its Applications in Moving Boundary

Jung Ae Kim · Keonhee Lee

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Abstract We study the existence and energy decay of solutions for the strongly damped nonlinear beam equation. We apply a method based on Nakao method to show that the solution decays exponentially, and to obtain precise estimates of the constants in the estimates. Finally, we discuss its applications in moving boundary.

Keywords Nonlinear beam equation · Moving boundary · Nakao method · Galerkin method

1 Introduction

In this paper, we investigate the existence and energy decay of solutions for the following strongly damped nonlinear beam equation:

$$\begin{aligned} &Ku'' + A^2u + \big(\alpha + M(\|A^{1/2}u\|^2)\big)Au + \delta Au' = 0, \\ &u(0) = u_0, \qquad u'(0) = u_1, \end{aligned} \tag{1.1}$$

where the operator A and the function $M(\cdot)$ satisfy some convenient assumptions, $\delta > 0$ is a constant and α is arbitrary real number. Moreover, we study the existence and uniqueness of strong solutions as well as the uniform decay of the energy to application of (1.1) over a non-cylindrical domain

$$\begin{aligned} &u_{tt} + \Delta^2u - \big(\alpha + M(\|\nabla u\|^2)\big)\Delta u + \Delta u_t = 0 \quad \text{in } \hat{Q}, \\ &u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \hat{\Sigma}, \\ &u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) \quad \text{in } \Omega_0, \end{aligned} \tag{1.2}$$

J.A. Kim (✉) · K. Lee
Department of Mathematics, Chungnam National University, Daejeon 305-764, Korea
e-mail: jakim@nims.re.kr

K. Lee
e-mail: khlee@math.cnu.ac.kr

where $\nu = \nu(\sigma, t)$ is the unit normal at $(\sigma, t) \in \hat{\Sigma}$ directed towards the exterior of $\hat{Q}$. If we denote by η the outer unit normal to the boundary Γ of Ω, we have, using a parametrization of Γ,

$$\nu(\sigma, t) = \frac{1}{\nu}(\eta(\xi), -\gamma'(t)\xi \cdot \eta(\xi)), \qquad \xi = \frac{\sigma}{\gamma(t)}, \quad \text{where } \nu = (1 + \gamma'(t)|\xi \cdot \eta(\xi)|^2)^{1/2}. \tag{1.3}$$

The transverse deflection of an extensible beam of length $L > 0$ whose ends are held at fixed distance apart as a physical model of (1.1) is written in the form of the hyperbolic equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial^2 u(x,t)}{\partial x^4} = \left(\alpha + \beta \int_0^L \left|\frac{\partial u(s,t)}{\partial s}\right|^2 ds\right) \frac{\partial^2 u(x,t)}{\partial x^2},$$

which has been proposed by Woinowsky and Krieger [18], where $u(x, t)$ is the deflection of the point x of the beam at the time t and $\alpha, \beta > 0$ are constants.

The abstract formulation as (1.1) was investigated by several authors [10–12, 14, 15]. In a fixed domain, it is well known that the energy of the system (1.2) also decays to zero, see [8, 13]. But in a moving domain, see [2, 5–7, 17], energy decay were more difficult to obtain than the result in fixed domain. And in this case, the transverse deflection $u(x, t)$ of a beam which changed its configuration at each instant of time increases its deformation, and hence increases its tension. Moreover, the horizontal movement of the boundary yields nonlinear terms involving derivatives in the space variable. See also Eisley [4] and Burgreen [3] for physics justification and background of the model. In this paper, we prove the existence, uniqueness and study asymptotic behavior for solutions for the problem (1.1) with $\delta = 1$ under some assumptions on A, K and $M(\cdot)$. The method applied in this paper is based on the Galerkin method for existence and the Nakao method for asymptotic behaviour of solutions. Also, our existence result of (1.2) on non-cylindrical domains will apply to the inverse transformation. This method was introduced by Rasso and Ughi [16] to study certain class of parabolic equations in non-cylindrical domains. Unlike the existing papers on stability for hyperbolic equations in non-cylindrical domain, we do not use the penalty method but work directly in our non-cylindrical domain $\hat{Q}$. From the physics point of view, system (1.2) describes the transverse deflection of a stretched beam fixed in moving boundary device. We use the standard notations which can be found in Bellman's book [1] and Lion's book [9]. The paper is organized as follows: In Sect. 2, we present some notations and assumptions which will be needed later. In Sect. 3, we will study the existence of solutions for the problem (1.1). In Sect. 4, we will study the exponential decay for solutions of the problem (1.1). Finally, in Sect. 5, we will show application of (1.1) in non-cylindrical domain.

2 Notations and Assumptions

In this section we prepare some notations and assumptions which will be needed in the proof of our result. Let H be a real Hilbert space with the inner product $(\cdot, \cdot)$ and the norm $\|\cdot\|$. And, in Sect. 5, we employ the usual notations for the standard functional spaces. The inner product and norm on $L^2(\Omega)$ and $L^2(\Gamma)$ are denoted by $(\cdot, \cdot)$, $\|\cdot\|$ and $(\cdot, \cdot)_\Gamma$, $\|\cdot\|_\Gamma$ respectively. We denote the Hilbert space $H(\Delta, \Omega) = \{u \in H^1(\Omega); \Delta u \in L^2(\Omega)\}$, provided with the norm

$$\|u\|_{H(\Delta,\Omega)} = (\|u\|^2_{H^1(\Omega)} + \|\Delta u\|^2)^{\frac{1}{2}},$$

where $H^1(\Omega)$ is the usual Sobolev space of first order. Now, we state the following hypotheses which will be assumed in this paper:

- (H.1) A is a linear operator in H, with the domain $D(A)$ dense in H and A is a self-adjoint and positive operator, i.e., there is a constant $k > 0$ such that $(Av, v) \geq k\|v\|^2$, for v in $D(A)$.
- (H.2) $M(\lambda)$ is a real C^1-class function with $M(\lambda) \geq m_0 + m_1\lambda$ for all $\lambda \geq 0$, $m_0 > 0$, $m_1 > 0$.
- (H.3) K is a linear and self-adjoint operator with domain H and $(Ku, u) > 0$ for all $u \in H$.

For priori estimates, we need the following Gronwall's inequality (see [1]):

Lemma 2.1 *If $k > 0$ and $c \geq 0$ are constants, then the inequality*

$$f(t) \leq k + c\int_0^t h(s)f(s)ds$$

implies that

$$f(t) \leq k\exp\left(c\int_0^t h(s)ds\right), \quad t \geq 0.$$

To prove the decay estimate of the energy, we use the following difference inequality by Nakao and its proof can be found in [11].

Lemma 2.2 *Let $E(t)$ be a nonincreasing and nonnegative function on $[0, T]$ such that*

$$E(t) \leq k_0(E(t) - E(t+1)) \quad \text{on } [0, T],$$

where k_0 is a positive constant. Then we have

$$E(t) \leq E(0)e^{-k_1[t-1]^+} \quad \text{on } [0, T], \quad \text{where } k_1 = \log\left(\frac{k_0}{k_0 - 1}\right).$$

3 Existence Results of (1.1)

In this section, we are going to show the existence of solution of problem (1.1) using Faedo-Galerkin's approximation. Considering the hypotheses (H.1)–(H.3) we have the following existence result:

Theorem 3.1 *Let α be arbitrary real number, u_0 belongs to $D(A^{k+1/2})$, u_1 belongs to $D(A^k)$ for all natural number k. Then there is a unique function $u(t)$, $0 \leq t < T$ for $0 < T < \infty$, such that $u(t)$ and $u'(t)$ belong to $C([0, T); D(A^k))$ for all natural number k and $u(t)$ satisfies*

$$Ku'' + A^2u + \big(\alpha + M(\|A^{1/2}u\|^2)\big)Au + Au' = 0 \quad \text{in } H, \tag{3.4}$$

$$u(0) = u_0, \qquad u'(0) = u_1. \tag{3.5}$$

Proof For simplicity of proof, we only consider $\delta = 1$ without loss of generality. To prove the existence we use the Galerkin method, taking for base the eigenvectors of the operator A. In fact, let $(w_\nu)_{\nu\in\mathbb{N}}$ be the sequence of eigenvectors of A corresponding to the sequence $(\lambda_\nu)_{\nu\in\mathbb{N}}$ of its eigenvalues. Let $V_m = [w_1, w_2, \ldots, w_m]$ be the m-dimensional subspace of

H generated by the m first eigenvectors of A. We define the approximate solutions $u_m(t)$ of (3.1) in the following way: $u_m(t) \in V_m$, i.e.,

$$u_m(t) = \sum_{i=1}^{m} g_{im}(t) w_i, \tag{3.6}$$

$$\begin{aligned}&(Ku''_m(t), v) + (Au_m(t), Av)\\ &\quad + \big(\alpha + M(\|A^{1/2}u_m(t)\|^2)\big)(Au_m(t), v) + (Au'_m(t), v) = 0,\end{aligned} \tag{3.7}$$

$$u_m(0) = u_{0m}, \qquad u'_m(0) = u_{1m}. \tag{3.8}$$

By u_{0m} and u_{1m} we represent, respectively, the partial sum of the m first terms in the eigenvectors expansion of the vectors u_0 and u_1. Systems (3.7) and (3.8) of ordinary differential equations, in the variable t, has a solution $u_m(t)$ in an interval $[0, t_m)$. In the next step we obtain a priori estimate for the solution $u_m(t)$, so that it can be extended outside $[0, t_m)$, to obtain one solution defined for all $t > 0$.

Step 1: First Priori Estimate

Taking $v = 2u'_m(t)$ in (3.7), we obtain

$$\begin{aligned}&\frac{d}{dt}\|K^{1/2}u'_m(t)\|^2 + \frac{d}{dt}\|Au_m(t)\|^2 + \alpha\frac{d}{dt}\|A^{1/2}u_m(t)\|^2\\ &\quad + M(\|A^{1/2}u_m(t)\|^2)\frac{d}{dt}\|A^{1/2}u_m(t)\|^2 + 2\|A^{1/2}u_m(t)\|^2 = 0.\end{aligned} \tag{3.9}$$

Integrating from 0 to t, with $0 \le t < t_m$, we obtain

$$\begin{aligned}&\|K^{1/2}u'_m(t)\|^2 + \|Au_m(t)\|^2 + \alpha\|A^{1/2}u_m(t)\|^2\\ &\quad + \int_0^t \left(M(\|A^{1/2}u_m(s)\|^2)\frac{d}{ds}\|A^{1/2}u_m(s)\|^2\right)ds + 2\int_0^t \|A^{1/2}u'_m(s)\|^2 ds\\ &= \|K^{1/2}u_{1m}\|^2 + \|Au_{0m}\|^2 + \alpha\|A^{1/2}u_{0m}\|^2.\end{aligned}$$

Let $\hat{M}$ be the primitive of M defined by $\hat{M}(\lambda) = \int_0^\lambda M(s)ds$. We have

$$\begin{aligned}&\|K^{1/2}u'_m(t)\|^2 + \|Au_m(t)\|^2 + \alpha\|A^{1/2}u_m(t)\|^2\\ &\quad + \hat{M}(\|A^{1/2}u_m(t)\|^2) + 2\int_0^t \|A^{1/2}u'_m(s)\|^2 ds\\ &= P_m,\end{aligned}$$

with $P_m = \|K^{1/2}u_{1m}\|^2 + \|Au_{0m}\|^2 + \alpha\|A^{1/2}u_{0m}\|^2 + \hat{M}(\|A^{1/2}u_{0m}\|^2)$ which converges to $P_0 = \|K^{1/2}u_1\|^2 + \|Au_0\|^2 + \alpha\|A^{1/2}u_0\|^2 + \hat{M}(\|A^{1/2}u_0\|^2)$.

$$\begin{aligned}&\|K^{1/2}u'_m(t)\|^2 + \|Au_m(t)\|^2 + \hat{M}(\|A^{1/2}u_m(t)\|^2) + 2\int_0^t \|A^{1/2}u'_m(s)\|^2 ds\\ &\quad \le R_m + |\alpha|\|A^{1/2}u_m(t)\|^2,\end{aligned} \tag{3.10}$$

where $R_m = \|K^{1/2}u_{1m}\|^2 + |\alpha|\|A^{1/2}u_{0m}\|^2 + \|Au_{0m}\|^2 + \hat{M}(\|A^{1/2}u_{0m}\|^2)$ which converges to $R_0 = \|K^{1/2}u_1\|^2 + |\alpha|\|A^{1/2}u_0\|^2 + \|Au_0\|^2 + \hat{M}(\|A^{1/2}u_0\|^2)$. Therefore, there is a constant

$C_0 > 0$, independent of m and greater than R_m such that (3.10) still holds, with R_m replaced by C_0.

Now, $M(\lambda) \geq m_0 + m_1\lambda$ implies $\hat{M}(\lambda) \geq m_0\lambda + (m_1/2)\lambda^2$. For $|\alpha|\sigma \leq |\alpha|^2/(2m_1) + (m_1/2)\sigma^2$, we have

$$\|K^{1/2}u_m(t)\|^2 + \|Au_m(t)\|^2 + m_0\|A^{1/2}u_m(t)\|^2 + 2\int_0^t \|A^{1/2}u_m'(s)\|^2 ds \leq C, \tag{3.11}$$

where $C = C_0 + |\alpha|^2/(2m_1)$, a constant independent of m.

$$\|K^{1/2}u_m'(t)\|^2 + \|Au_m(t)\|^2 + \|A^{1/2}u_m(t)\|^2 + \|A^{1/2}u_m'(t)\|^2 \leq C - 2\int_0^T \|A^{1/2}u_m'(s)\|^2 ds.$$

Then

$$\begin{aligned} &\|u_m'(t)\|^2 + \|Au_m(t)\|^2 + \|A^{1/2}u_m(t)\|^2 + \|A^{1/2}u_m'(t)\|^2 \\ &\quad \leq C + \int_0^T \big(\|u_m'(s)\|^2 + \|Au_m(s)\|^2 + \|A^{1/2}u_m(s)\|^2 + \|A^{1/2}u_m'(s)\|^2\big) ds. \end{aligned}$$

Hence, applying the Gronwall inequality, we obtain

$$\|u_m'(t)\|^2 + \|Au_m(t)\|^2 + \|A^{1/2}u_m(t)\|^2 + \|A^{1/2}u_m'(t)\|^2 \leq Ce^T. \tag{3.12}$$

In particular, since $\delta\|u_m(t)\|^2 \leq (Au_m(t), u_m(t))$, it follows that $u_m(t)$ remains bounded; hence, it can be extend to $[0, T)$. Therefore, (3.12) holds for all m and $t \in [0, T)$.

Step 2: Second Priori Estimate

Since V_m is generated by the m first eigenvectors of A, we can take in (3.7) $v = 2A^{2k}u_m'(t)$, for all natural numbers k. We obtain

$$\begin{aligned} &\frac{d}{dt}\|K^{1/2}A^k u_m'(t)\|^2 + \frac{d}{dt}\|A^{k+1}u_m(t)\|^2 + \alpha\frac{d}{dt}\|A^{k+1/2}u_m(t)\|^2 \\ &\quad + M(\|A^{1/2}u_m(t)\|^2)\frac{d}{dt}\|A^{k+1/2}u_m(t)\|^2 + 2\|A^{k+1/2}u_m'(t)\|^2 = 0. \end{aligned} \tag{3.13}$$

Let us set $\gamma_k(t) = \|K^{1/2}A^k u_m'(t)\|^2 + \|A^{k+1}u_m(t)\|^2$, $\beta_k(t) = \|A^{k+1/2}u_m(t)\|^2$ and $\mu(t) = M(\|A^{1/2}u_m(t)\|^2)$. Then (3.13) can be written as

$$\frac{d}{dt}\gamma_k(t) + \alpha\frac{d}{dt}\beta_k(t) + \mu(t)\frac{d}{dt}\beta_k(t) + 2\|A^{k+1/2}u_m'(t)\|^2 = 0. \tag{3.14}$$

If we set $h_k(t) = \gamma_k(t) + \alpha\beta_k(t) + \mu(t)\beta_k(t)$, then we obtain

$$h_k(t) - \alpha\beta_k(t) = \gamma_k(t) + \mu(t)\beta_k(t) \tag{3.15}$$

and hence $h_k(t) - \alpha\beta_k(t)$ is a positive function for $t \geq 0$. Derivating both sides of (3.15) with respect to t, we obtain

$$\frac{d}{dt}(h_k(t) - \alpha\beta_k(t)) = \frac{d}{dt}\gamma_k(t) + \mu(t)\frac{d}{dt}\beta_k(t) + \frac{d}{dt}\mu(t)\beta_k(t). \tag{3.16}$$

From (3.14) and (3.16) we can write

$$\frac{d}{dt}(h_k(t)-\alpha\beta_k(t)) = -\alpha\frac{d}{dt}\beta_k(t)+\frac{d}{dt}\mu(t)\beta_k(t)-2\|A^{k+1/2}u'_m(t)\|^2. \tag{3.17}$$

Using the definition of $\beta_k(t)$, finding its derivative with respect to t and by the Schwartz inequality, we have

$$\left|\frac{d}{dt}\beta_k(t)\right| \le \gamma_k(t). \tag{3.18}$$

From (3.15), (3.17) and (3.18) we find

$$\begin{aligned}\left|\frac{d}{dt}(h_k(t)-\alpha\beta_k(t))\right| &\le |\alpha|\gamma_k(t)+\left|\frac{d}{dt}\mu(t)\right|\beta(t)\\ &\le \left(|\alpha|+\frac{1}{\mu(t)}\left|\frac{d}{dt}\mu(t)\right|\right)(h_k(t)-\alpha\beta_k(t)).\end{aligned} \tag{3.19}$$

Since $\mu(t)=M(\|A^{1/2}u_m(t)\|^2)$, $M(\lambda)\in C^1[0,\infty)$, we have

$$\begin{aligned}\left|\frac{d}{dt}\mu(t)\right| &= |M'(\|A^{1/2}u_m(t)\|^2)||2(Au_m(t),u'_m(t))|\\ &\le |M'(\|A^{1/2}u_m(t)\|^2)|(\|u'_m(t)\|^2+\|Au_m(t)\|^2).\end{aligned}$$

By the first estimate (3.12) it follows that $|\frac{d}{dt}\mu(t)|\le C_1\max_{0\le s\le C_1}|M'(s)|$, where $C_1=Ce^T$.

Therefore, (3.19) can be written as

$$\left|\frac{d}{dt}(h_k(t)-\alpha\beta_k(t))\right| \le C_2(h_k(t)-\alpha\beta_k(t)), \tag{3.20}$$

where $C_2=|\alpha|+\frac{C_1}{m_0}\max_{0\le s\le C_1}|M'(s)|$.

From (3.20) and the Gronwall inequality (Lemma 2.1), we obtain

$$h_k(t)-\alpha\beta_k(t)\le (h_k(0)-\alpha\beta_k(0))\exp(C_2t)\quad \text{for all } t\ge 0. \tag{3.21}$$

If follows from (3.21) and according to assumptions on the initial data, that there exists a constant $C_3>0$, independent of m, such that

$$\|K^{1/2}A^k u'_m(t)\|^2+\|A^{k+1}u_m(t)\|^2+\|A^{k+1/2}u_m(t)\|^2\le C_3, \tag{3.22}$$

for all natural number k and all real number $t\ge 0$.

Step 3: Limit of Approximate Solutions

We prove, first of all, that the priori estimate (3.22) imply that sequences $(K^{1/2}A^k u'_m(t))_{m\in\mathbb{N}}$, $(A^{k+1/2}u_m(t))_{m\in\mathbb{N}}$, $(A^k u_m(t))_{m\in\mathbb{N}}$ are Cauchy sequences on $[0,T]$, for all $T>0$. Suppose that m_1 and m_2 are natural numbers such that $m_2\ge m_1$, $u_m(t)=\sum_{i=1}^m g_{im}(t)w_i$. Set $w(t)=u_{m_2}(t)-u_{m_1}(t)$, where $u_{m_1}(t)$ and $u_{m_2}(t)$ are two solutions of the approximate system (3.7). It follows that $w(t)$ satisfies the following three conditions:

$$\begin{aligned}
&(Kw''(t), v) + (Aw(t), Av) + \alpha(Aw(t), v) \\
&\qquad + M(\|A^{1/2}u_{m_2}(t)\|^2)(Aw(t), v) + (Aw'(t), v) \\
&\quad = \big(M(\|A^{1/2}u_{m_1}(t)\|^2) - M(\|A^{1/2}u_{m_2}(t)\|^2)\big)(Au_{m_1}, v); \qquad (3.23) \\
&w(0) = u_{m_2}(0) - u_{m_1}(0) = u_{0m_2} - u_{0m_1} \to 0 \quad \text{as } m_1, m_2 \to 0; \\
&w'(0) = u'_{m_2}(0) - u'_{m_1}(0) = u_{1m_2} - u_{1m_1} \to 0 \quad \text{as } m_1, m_2 \to 0.
\end{aligned}$$

Taking $v = 2A^{2k}w'(t)$ in (3.23), we obtain

$$\begin{aligned}
&\frac{d}{dt}\|K^{1/2}A^k w'(t)\|^2 + \frac{d}{dt}\|A^{k+1}w(t)\|^2 + \alpha\|A^{k+1/2}w(t)\|^2 \\
&\quad + M(\|A^{1/2}u_{m_2}(t)\|^2)\frac{d}{dt}\|A^{k+1/2}w(t)\|^2 + 2\|A^{k+1/2}w'(t)\|^2 \\
&= 2\big(M(\|A^{1/2}u_{m_1}(t)\|^2) - M(\|A^{1/2}u_{m_2}(t)\|^2)\big)(Au_{m_1}(t), A^{2k}w'(t)). \qquad (3.24)
\end{aligned}$$

Let us set

$$\begin{aligned}
&\gamma_k(t) = \|K^{1/2}A^k w'(t)\|^2 + \|A^{k+1}w(t)\|^2 \quad \text{and} \\
&\beta_k(t) = \|A^{k+1/2}w(t)\|^2, \qquad \mu_\nu(t) = M(\|A^{1/2}u_{m_\nu}(t)\|^2) \quad (\nu = 1, 2);
\end{aligned}$$

with this notation, the differential equation (3.24) can be written as

$$\begin{aligned}
&\frac{d}{dt}\gamma_k(t) + \alpha\frac{d}{dt}\beta_k(t) + \mu_2(t)\frac{d}{dt}\beta_k(t) \\
&\quad = 2(\mu_1(t) - \mu_2(t))(A^k u_{m_1}(t), A^k w'(t)) - 2\|A^{k+1/2}w'(t)\|^2. \qquad (3.25)
\end{aligned}$$

We define $h_k(t)$ by $h_k(t) = \gamma_k(t) + \alpha\beta_k(t) + \mu_2(t)\beta_k(t)$.

Taking the derivative of the last equation and using (3.25), we obtain

$$\begin{aligned}
\frac{d}{dt}(h_k(t) - \alpha\beta_k(t)) &= -\alpha\frac{d}{dt}\beta_k(t) + \frac{d}{dt}\mu_2(t)\beta_k(t) \\
&\quad + 2(\mu_1(t) - \mu_2(t))(A^k u_{m_1}(t), A^k w'(t)) - 2\|A^{k+1/2}w'(t)\|^2.
\end{aligned}$$

By the definition of $\mu_\nu(t)$ and (3.22), we have $|\mu_2(t) - \mu_1(t)| \le C_4\|A^{1/2}w(t)\|$ and we also have $|(A^k u_{m_1}(t), A^k w'(t))| \le C_3\|A^k w'(t)\|$.

Therefore we obtain by (3.18),

$$\begin{aligned}
&\left|\frac{d}{dt}(h_k(t) - \alpha\beta_k(t))\right| \\
&\quad \le |\alpha|\gamma_k(t) + \left|\frac{d}{dt}\mu_2(t)\right|\beta_k(t) + 2C_3C_4|A^{1/2}w(t)||A^k w'(t)| \\
&\quad \le |\alpha|\gamma_k(t) + \left|\frac{d}{dt}\mu_2(t)\right|\beta_k(t) + K|A^{1/2}w(t)|^2 + K|A^k w'(t)|^2, \qquad (3.26)
\end{aligned}$$

where $K = C_3C_4$.

Since $\gamma_k(t) \le h_k(t) - \alpha\beta_k(t)$, $\beta_k(t) \le (h_k(t) - \alpha\beta_k(t))/\mu_2(t)$ and $|A^k w'(t)|^2 \le \gamma_k(t)$, we obtain the following form:

$$\begin{aligned}
&\left|\frac{d}{dt}(h_k(t) - \alpha\beta_k(t))\right| \\
&\quad\le |\alpha|(h_k(t) - \alpha\beta_k(t)) + (h_k(t) - \alpha\beta_k(t))\left|\frac{d}{dt}\mu_2(t)\right|\frac{1}{\mu_2(t)} \\
&\qquad + K(h_k(t) - \alpha\beta_k(t)) + K\beta_0(t) \\
&\quad\le K_0(h_k(t) - \alpha\beta_k(t)) + K\beta_0(t),
\end{aligned} \tag{3.27}$$

where $K_0 = |\alpha| + M + K$.

If we set $k = 0$ in (3.27), we obtain

$$\begin{aligned}
\left|\frac{d}{dt}(h_0(t) - \alpha\beta_0(t))\right| &\le K_0(h_0(t) - \alpha\beta_0(t)) + K\beta_0(t) \\
&\le (K_0 + K/m_0)(h_0(t) - \alpha\beta_0(t)).
\end{aligned}$$

Thus we have

$$h_0(t) - \alpha\beta_0(t) \le (h_0(0) - \alpha\beta_0(0))\exp(K_0 + K/m_0)t.$$

Since $h_0(t) - \alpha\beta_0(t) = \gamma_0(t) + \mu_2(t)\beta_0(t) \ge \gamma_0(t) + m_0\beta_0(t)$, we obtain

$$\beta_0(t) \le (\gamma_0(0) + K\beta_0(0))\exp(K_0 + K/m_0)T/m_0 = K_3,$$

where K_2 depends only on the constant $C_1 = Ce^T$ of (3.12). Therefore we obtain the following inequality

$$\frac{d}{dt}(h_k(t) - \alpha\beta_k(t)) \le K_0(h_k(t) - \alpha\beta_k(t)) + KK_3.$$

Hence

$$h_k(t) - \alpha\beta_k(t) \le (h_k(0) - \alpha\beta_k(0) + KK_3T)e^{TK_0}.$$

From this inequality, we obtain

$$\gamma_k(t) + m_0\beta_k(t) \le (\gamma_k(0) + M(\|A^{1/2}u_{0m_2}\|^2)\beta_k(0))e^{TK_0} + K_4(\gamma_k(0) + K_2\beta_0(0)),$$

where $K_4 = KT\exp(K_0 + K/m_0)T/m_0e^{TK_0}$ for all $0 \le t \le T$.

By the definition of $\gamma_k(t)$ and $\beta_k(t)$, and the last inequality we obtain

$$\begin{aligned}
&\|K^{1/2}A^k u'_{m_2}(t) - K^{1/2}A^k u'_{m_1}(t)\|^2 + \|A^{k+1}u_{m_2}(t) - A^{k+1}u_{m_1}(t)\|^2 \\
&\quad + m_0\|A^{k+1/2}u_{m_2}(t) - A^{k+1/2}u_{m_1}(t)\|^2 \\
&\le \big(\|K^{1/2}A^k u_{1m_2} - K^{1/2}A^k u_{1m_1}\|^2 + \|A^{k+1}u_{0m_2} - A^{k+1}u_{0m_1}\|^2 \\
&\quad + M(\|A^{1/2}u_{0m_2}\|^2)\|A^{k+1/2}u_{0m_2} - A^{k+1/2}u_{0m_1}\|^2\big)\exp(TK_0) \\
&\quad + K_4\big(\|K^{1/2}u_{1m_2} - K^{1/2}u_{1m_1}\|^2 + \|Au_{0m_2} - Au_{0m_1}\|^2 \\
&\quad + K_2\|A^{1/2}u_{0m_2} - A^{1/2}u_{0m_1}\|^2\big).
\end{aligned} \tag{3.28}$$

It follows that the maximum on $[0,T]$ of each term in the left-hand side of (3.28) converges to zero as m_1, m_2 go to infinity. Therefore, we conclude that the sequences $(K^{1/2}A^k u_m'(t))$, $(A^{k+1}u_m(t))$, $(A^{k+1/2}u_m(t))$ are Cauchy sequences of $C([0,T],H)$, hence they are convergent for all $k \in \mathbb{N}$. For $k=0$ we obtain the existence of a vector $u(t)$ such that $(u_m(t))_{m\in\mathbb{N}}$ converges to $u(t)$ in $C([0,T],H)$. We also conclude that $(A^{1/2}u_m(t))$ converges to $(A^{1/2}u(t))$ in $C([0,T],H)$ and M is continuous so that we can pass to the limit in $M(\|A^{1/2}u_m(t)\|^2)$. From those convergence, it follows that u such that

$$\frac{d}{dt}(Ku'(t),v) + (Au(t),Av) + \big(\alpha + M(\|A^{1/2}u(t)\|^2)\big)(Au(t),v) + (Au'(t),v) = 0,$$

for all $v \in H$, that is, u is a solution of (3.7). From the uniform convergence of $(u_m(t))$ and $(u_m'(t))$ in $C([0,T],H)$, it follows that the limit u satisfies the initial conditions.

Step 4: Uniqueness of the Weak Solution

Suppose u and $\bar{u}$ are two solutions for the problem (3.4) and (3.5). Set $w = u - \bar{u}$. Then we have

$$\begin{aligned} &Kw''(t) + A^2w(t) + \alpha Aw(t) + M(\|A^{1/2}u(t)\|^2)Aw(t) + Aw'(t) \\ &\quad = \big(M(\|A^{1/2}\bar{u}(t)\|^2) - M(\|A^{1/2}u(t)\|^2)\big)A\bar{u}(t), \qquad (3.29) \\ &w(0) = 0, \qquad w'(0) = 0. \qquad (3.30) \end{aligned}$$

Multiplying (3.29) by $2w'$, we obtain

$$\begin{aligned} &\frac{d}{dt}\|K^{1/2}w'(t)\|^2 + \frac{d}{dt}\|Aw(t)\|^2 + \alpha\frac{d}{dt}\|A^{1/2}w(t)\|^2 \\ &\quad + M(\|A^{1/2}u(t)\|^2)\frac{d}{dt}\|A^{1/2}w(t)\|^2 + 2\|A^{1/2}w'(t)\|^2 \\ &= 2\big(M(\|A^{1/2}\bar{u}(t)\|^2) - M(\|A^{1/2}u(t)\|^2)\big)(A\bar{u}, w'(t)). \qquad (3.31) \end{aligned}$$

Integrating (3.31) from 0 to t, we obtain

$$\begin{aligned} &\|K^{1/2}w'(t)\|^2 + \|Aw(t)\|^2 + \alpha\|A^{1/2}w(t)\|^2 \\ &\quad + \int_0^t M(\|A^{1/2}u(s)\|^2)\frac{d}{ds}\|A^{1/2}w(s)\|^2 ds + 2\int_0^t \|A^{1/2}w'(s)\|^2 ds \\ &= 2\int_0^t \big(M(\|A^{1/2}\bar{u}(s)\|^2) - M(\|A^{1/2}u(s)\|^2)\big)(A\bar{u}(s), w'(s))ds. \end{aligned}$$

Whence

$$\begin{aligned} &\|K^{1/2}w'(t)\|^2 + \|Aw(t)\|^2 + \alpha\|A^{1/2}w(t)\|^2 + M(\|A^{1/2}u(t)\|^2)\|A^{1/2}w(t)\|^2 \\ &= \int_0^t \|A^{1/2}w(s)\|^2\frac{d}{ds}M(\|A^{1/2}u(s)\|^2)ds - 2\int_0^t \|A^{1/2}w'(s)\|^2 ds \\ &\quad + 2\int_0^t \big(M(\|A^{1/2}\bar{u}(s)\|^2) - M(\|A^{1/2}u(s)\|^2)\big)(A\bar{u}(s), w'(s))ds. \end{aligned}$$

Nothing that $M(\|A^{1/2}u(t)\|^2) \geq m_0$, $|M(\|A^{1/2}\bar{u}\|^2) - M(\|A^{1/2}u\|^2)| \leq C_0\|A^{1/2}w(t)\|$, and $\frac{d}{dt}M(\|A^{1/2}u(t)\|^2) = 2M'(\|A^{1/2}u(t)\|^2)(Au(t), u'(t))$ is bounded, we obtain

$$\begin{aligned}&\|w'(t)\|^2 + \|Aw(t)\|^2 + \|A^{1/2}w(t)\|^2 \\ &\quad\leq C\int_0^t \big(\|w'(s)\|^2 + \|Aw(s)\|^2 + \|A^{1/2}w(s)\|^2\big)ds. \end{aligned} \tag{3.32}$$

From (3.32) and the Gronwall inequality (Lemma 2.1), we obtain $w(t) = 0$ in $[0, T]$, that is, $u = \bar{u}$. The proof of Theorem 3.1 is completed. □

4 Energy Decay Result of (1.1)

In this section we prove the exponential decay of solutions of problem (1.1) by using Nakao method. Now we are in a position to show the stability problem of (1.1):

Theorem 4.1 *Under hypotheses of Theorem* 3.1, *the solution of the problem* (3.4)–(3.5) *satisfies*:

$$\|K^{1/2}u'(t)\|^2 + \|Au(t)\|^2 + \alpha\|A^{1/2}u(t)\|^2 + \int_t^{t+1} \|A^{1/2}u'(s)\|^2 ds \leq \alpha_1 e^{-\alpha_2 t} \tag{4.33}$$

for all $t \geq 1$, *where* α_i, $i = 1, 2$, *are constants.*

Proof In order to use Nakao's method, we consider the approximated equation

$$\begin{aligned}&(Ku_m''(t), w) + (Au_m(t), Aw) \\ &\quad + \big(\alpha + M(\|A^{1/2}u_m(t)\|^2)\big)(Au_m(t), w) + (Au_m'(t), w) = 0, \end{aligned} \tag{4.34}$$

for all $w \in V_m$.

Letting $w = u_m'(t)$ in (4.34), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|K^{1/2}u_m'(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|Au_m(t)\|^2 \\ &\quad + \big(\alpha + M(\|A^{1/2}u_m(t)\|^2)\big)\frac{1}{2}\frac{d}{dt}\|A^{1/2}u_m(t)\|^2 + \|A^{1/2}u_m'(t)\|^2 = 0. \end{aligned} \tag{4.35}$$

Integrating (4.35) from 0 to t, we obtain

$$E(t) + \int_0^t \|A^{1/2}u_m'(s)\|^2 ds = E(0), \tag{4.36}$$

where $E(t) = 1/2(\|K^{1/2}u_m'(t)\|^2 + \|Au_m(t)\|^2 + \alpha\|A^{1/2}u_m(t)\|^2 + \hat{M}(\|A^{1/2}u_m(t)\|^2))$. From (3.12), we also obtain

$$\|K^{1/2}u_m'(t)\|^2 + \|Au_m(t)\|^2 + \|A^{1/2}u_m(t)\|^2 + \|A^{1/2}u_m'(t)\|^2 \leq C_1, \tag{4.37}$$

where $C_1 = Ce^T$. Integrating (4.35) from τ_1 to τ_2, $0 < \tau_1 < \tau_2$, we get

$$E(\tau_2) + \int_{\tau_1}^{\tau_2} \|A^{1/2}u_m'(s)\|^2 ds = E(\tau_1), \tag{4.38}$$

for all $t > 0$. Thus we have

$$\int_t^{t+1} \|A^{1/2}u'_m(s)\|^2 ds = E(t) - E(t+1) = F(t)^2. \tag{4.39}$$

Therefore, there exists two points $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$ such that

$$\|A^{1/2}u'_m(t_i)\| \le 2F(t) \quad (i = 1, 2). \tag{4.40}$$

Letting $w = u_m(t)$ in (4.34) and integrating the equation from t_1 to t_2 and $M(\|A^{1/2}u_m(t)\|^2) \ge m_0$, we obtain

$$\begin{aligned}
&(1 + C(\alpha + m_0)) \int_{t_1}^{t_2} \|Au_m(s)\|^2 ds \\
&\quad \le (Ku'_m(t_1), u_m(t_1)) - (Ku'_m(t_2), u_m(t_2)) \\
&\qquad + \int_{t_1}^{t_2} (\|K^{1/2}u'_m(s)\|^2 - (Au'_m(s), u_m(s)))ds \\
&\quad \le (K_0^2 + C'^2) \int_{t_1}^{t_2} \|u'_m(t)\|^2 ds + C^2 \int_{t_1}^{t_2} \|Au_m(s)\|^2 ds \\
&\qquad + CK_0 \operatorname*{ess\,sup}_{s\in[t,t+1]} \|Au_m(s)\|(\|u'_m(t_1)\| + \|u'_m(t_2)\|),
\end{aligned}$$

where $C > 0$ is a constant such that $\|u_m(t)\| \le C\|Au_m(t)\|$, $\|Ku'_m(t_i)\| \le K_0\|u'_m(t_i)\|$ $(i = 1, 2)$, and $\|Au'_m(t)\| \le C'\|u'_m(t)\|$. From (4.38) and (4.40), it follow that

$$\begin{aligned}
&(1 + C(\alpha + m_0) - C^2) \int_{t_1}^{t_2} \|Au_m(s)\|^2 ds \\
&\quad \le C_1\Big(F(t)^2 + \operatorname*{ess\,sup}_{s\in[t,t+1]} \|Au_m(s)\|F(t)\Big),
\end{aligned}$$

where $C_1 = \max(4CK_0, K_0^2 + C'^2)$.

Whence

$$\int_{t_1}^{t_2} \|Au_m(s)\|^2 ds \le C_2\Big(F(t)^2 + \operatorname*{ess\,sup}_{s\in[t,t+1]} \|Au_m(s)\|F(t)\Big) = G(t)^2. \tag{4.41}$$

It follows from (4.39) and (4.41) that

$$\int_{t_1}^{t_2} \big(\|A^{1/2}u'_m(t)\|^2 + \|Au_m(s)\|^2\big) ds \le F(t)^2 + G(t)^2.$$

Hence there exist $t^* \in [t_1, t_2]$ such that

$$\|A^{1/2}u'_m(t^*)\|^2 + \|Au_m(t^*)\|^2 \le 2(F(t)^2 + G(t)^2). \tag{4.42}$$

Noting now that

$$\hat{M}(\|A^{1/2}u_m(t^*)\|^2) \le n_0\|A^{1/2}u_m(t^*)\|^2,$$

where $n_0 = \max_{0\le s\le \|A^{1/2}u_m(t^*)\|^2} M(s)$, we get

$$\hat{M}(\|A^{1/2}u_m(t^*)\|^2) \le C_3(F(t)^2 + G(t)^2). \tag{4.43}$$

From (4.36), (4.42) and (4.43), it follows that

$$E(t^*) \le C_4(F(t)^2 + G(t)^2). \tag{4.44}$$

Now, by (4.38), (4.39) and (4.44), we have

$$\begin{aligned}\operatorname*{ess\,sup}_{s\in[t,t+1]} E(s) &\le E(t^*) + \int_{t_1}^{t_2} \|A^{1/2}u_m'(s)\|^2 ds\\ &\le C_5 F(t)^2 + C_4 \operatorname*{ess\,sup}_{t\in[t,t+1]} \|Au_m(s)\| F(t)\\ &\le C_5 F(t)^2 + \frac{1}{2}\operatorname*{ess\,sup}_{s\in[t,t+1]} E(s).\end{aligned}$$

Therefore $E(t) \le C_7(E(t) - E(t+1))$ with C_7 a positive constant. We suppose that $C_7 > 1$ then $\gamma = 1/C_7 < 1$. So $E(t+1) \le (1-\gamma)E(t)$. Let $t \ge 1$ and we consider an integer n such that $n \le t \le n+1$. Then

$$E(t) \le (1-\gamma)E(t-1) \le (1-\gamma)^n E(t-n) \le m_1(1-\gamma)^n, \tag{4.45}$$

where $m_1 = \operatorname{ess\,sup}_{t\in[0,1]} E(t) < \infty$. Then $E(t) \le \gamma_1 e^{t\log(1-\gamma)} = \gamma_1 e^{-\alpha_2 t}$, $t \ge 1$, where $\gamma_1 = \frac{m_1}{1-\gamma}$ and $\alpha_2 = -\log(1-\gamma) > 0$. Whence

$$\begin{aligned}&\|K^{1/2}u_m'(t)\|^2 + \|Au_m(t)\|^2 + \alpha\|A^{1/2}u_m(t)\|^2 + \hat{M}(\|A^{1/2}u_m(t)\|^2)\\ &\quad\le 2\gamma_1 e^{-\alpha_2 t}\end{aligned} \tag{4.46}$$

for all $t \ge 1$.

Therefore, by $M(\lambda) \ge m_0$, (4.38) and (4.46) it follows that

$$\begin{aligned}&\|K^{1/2}u_m'(t)\|^2 + \|Au_m(t)\|^2 + \alpha\|A^{1/2}u_m(t)\|^2\\ &\quad + \int_t^{t+1} \|A^{1/2}u_m'(s)\|^2 ds \le \alpha_1 e^{-\alpha_2 t}\end{aligned} \tag{4.47}$$

for all $t \le 1$ with $\alpha_i > 0$ $(i = 1, 2)$.

By the same way in proof of Theorem 3.1, we obtain

$$\|K^{1/2}u'(t)\|^2 + \|Au(t)\|^2 + \alpha\|A^{1/2}u(t)\|^2 + \int_t^{t+1} \|A^{1/2}u'(s)\|^2 ds \le \alpha_1 e^{-\alpha_1 t}$$

for all $t \ge 1$, which completes our proof. □

5 Existence and Stability Result of (1.2)

In this section we prove the uniform decay, as time goes to infinity, of regular solutions for a nonlinear beam equation with viscous damping $u_{tt} + \Delta^2 u - (\alpha + M(\|\nabla u\|^2))\Delta u - \Delta u_t = 0$ in a non-cylindrical domain of $R^{n+1} (n \ge 1)$ under suitable hypothesis.

Let Ω be an open bounded domain of $\mathbb{R}^n$ containing the origin and having C^2 boundary and $\gamma : [0, \infty) \to R$ a continuously differential function. Consider the family of subdomains $\{\Omega_t\}_{0 \le t < \infty}$ of $\mathbb{R}^n$ given by

$$\Omega_t = T(\Omega), \qquad T : y \in \Omega \to x = \gamma(t)y,$$

whose boundaries are denoted by Γ_t, and let $\hat{Q}$ be the non-cylindrical domain of R^{n+1} given by

$$\hat{Q} = \bigcup_{0 \le t < \infty} \Omega_t \times \{t\} \quad \text{with lateral boundary } \hat{\Sigma} = \bigcup_{0 \le t < \infty} \Gamma_t \times \{t\}.$$

Let $\phi = 0$ be a parametrization of a part $\bigcup$ of Γ, $\bigcup$ containing $\xi = \sigma/\gamma(t)$. The parametrization of a part $\bigcup$ of $\hat{\Sigma}$ is $\psi(\sigma, t) = \phi(\sigma/\gamma(t)) = \phi(\xi) = 0$. We have

$$\nabla \psi(\sigma, t) = \frac{1}{\gamma(t)} (\nabla \phi(\xi), -\gamma'(t)\xi \cdot \nabla \phi(\xi)). \tag{5.48}$$

From this and observing that $\eta(\xi) = \nabla \phi(\xi)/|\nabla \phi(\xi)|$, expression (1.3) follows. Let $\bar{\nu}(\cdot, t)$ be the x-component of unit normal $\nu(\cdot, \cdot)$, $|\bar{\nu}| \le 1$. Then by relation (1.3), one has

$$\bar{\nu}(\sigma, t) = \eta\left(\frac{\sigma}{\gamma(t)}\right). \tag{5.49}$$

The method we use to prove the result of existence and uniqueness is based on transforming our problem into another initial boundary value problem defined over a cylindrical domain whose sections are not time dependent. This is done using a suitable change of variable. Then we show the existence and uniqueness for this new problem. Our existence result on domains with moving boundary will follow by using the inverse transformation, that is, by using the diffeomorphism.

$$\tau : \hat{Q} \to Q, \qquad (x, t) \in \Omega_t \to (y, t) = \left(\frac{x}{\gamma(t)}, t\right) \tag{5.50}$$

and $\tau^{-1} : Q \to \hat{Q}$ defined by

$$\tau^{-1}(y, t) = (x, t) = (\gamma(t)y, t). \tag{5.51}$$

Denoting by v the function

$$v(y, t) = u \circ \tau^{-1}(y, t) = u(\gamma(t)y, t), \tag{5.52}$$

the initial boundary value problem (1.2) becomes

$$\begin{aligned} & v_{tt} + \gamma^{-4}\Delta^2 v - \gamma^{-2}(\alpha + M(\gamma^{n-2}\|\nabla v\|^2))\Delta v + \gamma^{-2}\Delta v_t \\ & \quad - A(t)v + a_1 \cdot \nabla \partial_t v + a_2 \cdot \nabla v = 0 \quad \text{in } \Omega \times \mathbb{R}^+, \\ & v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \Gamma \times \mathbb{R}^+, \\ & v(y, 0) = v_0(y), \qquad v_t(y, 0) = v_1(y) \quad \text{in } \Omega, \end{aligned} \tag{5.53}$$

where

$$A(t)v = \sum_{i,j=1}^{n} \partial_{y_i}((a_{ij}\partial_{y_j} + \partial_{y_i} a'_{ij}\partial_{y_j})v) \tag{5.54}$$

$$a_{ij}(y,t) = -(\gamma'\gamma^{-1})^2 y_i y_j, \qquad a'_{ij}(y,t) = -(\gamma'\gamma^{-3}) y_i y_j \quad (i,j = 1,2,\dots,n), \tag{5.55}$$

$$a_1(y,t) = -2\gamma'\gamma^{-1} y, \qquad a_2(y,t) = -\gamma^{-2} y\big(\gamma''\gamma + (n-1)(\gamma')^2\big). \tag{5.56}$$

To show the existence of strong solution, we will use the following hypotheses:

$$\gamma' \le 0 \quad \text{if } n > 2, \qquad \gamma' \ge 0 \quad \text{if } n \le 2, \tag{5.57}$$

$$\gamma \in L^\infty(0,\infty), \qquad \inf_{0\le t<\infty} \gamma(t) = \gamma_0 > 0, \tag{5.58}$$

$$\gamma' \in W^{2,\infty}(0,\infty) \cap W^{2,1}(0,\infty). \tag{5.59}$$

Note that assumption (5.57) means that $\hat{Q}$ is decreasing if $n > 2$ and increasing if $n \le 2$ in the sense that when $t > t'$ and $n > 2$, then the projection of $\Omega_{t'}$ on the subspace $t = 0$ contains the projection of Ω_t on the same subspace and contrary in the case $n \le 2$. Concerning the function $M \in C^1[0,\infty)$, we assume that

$$M(\tau) \ge -m_0, \qquad M(\tau)\tau \ge \hat{M}(\tau), \quad \forall \tau \ge 0, \tag{5.60}$$

where $\hat{M}(\tau) = \int_0^\tau M(s)ds$ and

$$\Delta^2 w = \lambda_1 w \quad \text{in } \Omega, \qquad w = \frac{\partial w}{\partial \eta} = 0 \quad \text{in } \Gamma. \tag{5.61}$$

Now, we will study the existence and regularity of solutions for the system (1.2). The well posedness of system (5.53) is given by the following theorem.

Theorem 5.1 *Take* $v_0 \in H_0^2(\Omega) \cap H^4(\Omega)$, $v_1 \in H_0^2(\Omega)$ *and suppose that assumptions* (5.57)–(5.60) *hold. Then there exists a unique solution* v *of the problem* (5.53) *satisfying*

$$\begin{aligned} &v \in L^\infty(0,\infty : H_0^2(\Omega) \cap H^4(\Omega)), \qquad v_t \in L^\infty(0,\infty : H_0^2(\Omega)),\\ &v_{tt} \in L^\infty(0,\infty : L^2(\Omega)). \end{aligned} \tag{5.62}$$

Proof We denote by B the operator

$$Bw = \Delta^2 w, \qquad D(B) = H_0^2(\Omega) \cap H^4(\Omega). \tag{5.63}$$

It is well known that B is a positive self-adjoint operator in the Hilbert space $L^2(\Omega)$ for which there exist sequences $\{w_n\}_{n\in\mathbb{N}}$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ of eigenfunctions and eigenvalues of B such that the set of linear combinations of $\{w_n\}_{n\in\mathbb{N}}$ is dense in $D(B)$ and $\lambda_1 < \lambda_2 \le \cdots \le \lambda_n \to \infty$ as $n \to \infty$. We denote by V_m the space generated by $w_1, \dots, w_m$. Standard results on ordinary differential equations imply the existence of a local solution v_m of the form

$$v^m(t) = \sum_{j=1}^{m} g_{jm}(t) w_j, \tag{5.64}$$

to the system

$$\begin{aligned}&\big(v_{tt}^m+\gamma^{-4}\Delta^2 v^m-\gamma^{-2}\big(\alpha+M(\gamma^{n-2}\|\nabla v^m\|^2)\big)\Delta v^m+\gamma^{-2}\Delta v_t^m\\&\quad -A(t)v^m+a_1\cdot\nabla\partial_t v^m+a_2\cdot\nabla v^m,w_j\big)=0,\quad w_j\in V_m,\quad j=1,2,\ldots,m, \quad (5.65)\\&v^m(y,0)=v_0^m,\qquad v_t^m(y,0)=v_1^m. \quad (5.66)\end{aligned}$$

The extension of these solutions to the interval $[0,\infty)$ is a consequence of the first estimate which we are going to prove below. Applying similar method step 1 and step 2 of Theorem 3.1, we obtain the following estimates:

$$\|v_t^m(t)\|^2+\|\nabla v^m(t)\|^2+\|\Delta v^m(t)\|^2+\|\nabla v_t^m(t)\|^2\le L_1, \tag{5.67}$$

$$\|\Delta v_t^m(t)\|^2+\|\Delta^2 v^m(t)\|^2+\|\nabla\Delta v^m(t)\|^2\le L_2. \tag{5.68}$$

First of all, it easy to see from (5.65) that

$$\|v_{tt}^m(0)\|^2\le L_3.$$

Next, differentiating (5.65) with respect to the time, multiplying by $v''(t)$, and using similar arguments as above estimates, we obtain, after some calculations and taking into account above estimates,

$$\frac{1}{2}\frac{d}{dt}E_m(t)+\|\nabla v_{tt}(t)\|^2\le L_4(|\gamma'|+|\gamma''|)\|v_t^m(t)\|^2+L_4(|\gamma'|+|\gamma''|)E_m(t), \tag{5.69}$$

where

$$E_m(t)=\|v_{tt}^m(t)\|^2+\gamma^{-2}M(\gamma^{n-2}\|\nabla v^m(t)\|^2)\|\nabla v_t^m(t)\|^2. \tag{5.70}$$

Using Gronwall's lemma, we get

$$E_m(t)+\int_0^t\|\nabla v_{tt}(s)\|^2ds\le L_5. \tag{5.71}$$

From estimates (5.67)–(5.71), we have that v satisfies

$$\begin{aligned}&v^m\to v\quad \text{weakly star in } L^\infty(0,\infty:H_0^2(\Omega)\cap H^4(\Omega)),\\&v_t^m\to v_t\quad \text{weakly star in } L^\infty(0,\infty:H_0^2(\Omega)),\\&v_{tt}^m\to v_{tt}\quad \text{weakly star in } L^\infty(0,\infty:L^2(\Omega)).\end{aligned}$$

By compactness results $H_0^2(\Omega)\cap H^4(\Omega)\hookrightarrow H_0^1(\Omega)$ we can get a sequence such that

$$v^m\to v\quad \text{strongly in } L^2(0,\infty;H_0^1(\Omega)).$$

Moreover, since $M\in C^1[0,\infty)$ and ∇v^m is bounded in $L^\infty(0,\infty;L^2(\Omega))\cap L^2(0,\infty;L^2(\Omega))$, we have

$$\int_0^t\big|M\big(\gamma^{n-2}\|\nabla v^m\|^2\big)-M\big(\gamma^{n-2}\|\nabla v\|^2\big)\big|ds\le C\int_0^t\|v^m-v\|_{H_0^1(\Omega)}^2ds,$$

which is

$$M\big(\gamma^{n-2}\|\nabla v^m\|^2\big)(\Delta v^m,w_j)\to M(\gamma^{n-2}\|\nabla v\|^2)(\Delta v,w_j). \tag{5.72}$$

Letting $m \to \infty$ in (5.65), we conclude that

$$\begin{aligned} & v_{tt} + \gamma^{-4}\Delta^2 v - \gamma^{-2}\big(\alpha + M(\gamma^{n-2}\|\nabla v\|^2)\big)\Delta v + \gamma^{-2}\Delta v_t \\ & \quad - A(t)v + a_1 \cdot \nabla\partial_t v + a_2 \cdot \nabla v = 0 \quad \text{in } L^\infty(0,\infty; L^2(\Omega)). \end{aligned} \tag{5.73}$$

To prove the uniqueness of solutions of the problem (5.73), we use the method of the energy introduced by Lions [9], coupled with Gronwall's inequality and the hypotheses introduced in the paper about the functions M, g and the obtained estimates. □

To show the existence in noncylindrical domains, we return to our original problem in the noncylindrical domains by using the change variable given in (5.52) by $\tau^{-1}(y,t) = (x,t)$, $(x,t) \in \widehat{Q}$. Let v be the solution obtained from Theorem 5.1 and u defined by (5.52), then u belong to the classes

$$u \in L^\infty(0,\infty : H_0^2(\Omega_t) \cap H^4(\Omega_t)), \tag{5.74}$$

$$u_t \in L^\infty(0,\infty : H_0^2(\Omega_t)), \tag{5.75}$$

$$u_{tt} \in L^\infty(0,\infty : L^2(\Omega_t)). \tag{5.76}$$

Denoting that

$$u(x,t) = v(y,t) = (v \circ \tau)(x,t),$$

then from (5.52) follows that

$$u_{tt} + \Delta^2 u - \big(\alpha + M(\|\nabla u\|^2)\big)\Delta u - \Delta u_t = 0 \quad \text{in } L^\infty(0,\infty; L^2(\Omega_t)). \tag{5.77}$$

By uniqueness result of Theorem 5.1, we have $u_1 = u_2$, where u_1, u_2 are two solutions obtained through the diffeomorphism τ given by (5.50). Therefore, we have the following theorem:

Theorem 5.2 *Take $u_0 \in H_0^2(\Omega_0) \cap H^4(\Omega_0)$, $u_1 \in H_0^1(\Omega_0)$ and suppose that assumptions* (5.57) *and* (5.60) *hold. Then there exists a unique solution u of the* (1.2) *satisfying* (5.74) *and* (5.77).

Finally, we show that the solution of system (1.2) decays exponentially. Additionally, we assume that the function $\gamma(\cdot)$ satisfies the conditions

$$\gamma' \le 0, \qquad t \ge 0, \qquad n > 2, \tag{5.78}$$

$$0 < \max_{0 \le t < \infty} |\gamma'(t)| \le \frac{1}{d}, \tag{5.79}$$

where $d = \operatorname{diam}(\Omega)$. The condition (5.78) implies that our domain are "time-like" in the sense that $|\underline{\nu}| < |\overline{\nu}|$, where $\underline{\nu}$ and $\overline{\nu}$ denote the x-component and t-component of the outer unit normal of $\widehat{\Sigma}$.

First of all, we will prove the following lemmas that will be used in the sequel.

Lemma 5.1 *Let $M(\cdot,\cdot)$ be the smooth function defined in $\Omega_t \times [0,\infty)$. Then,*

$$\frac{d}{dt}\int_{\Omega_t} M(x,t)dx = \int_{\Omega_t} \frac{d}{dt} M(x,t)dx + \frac{\gamma'}{\gamma}\int_{\Gamma_t} M(x,t)(x \cdot \overline{\nu})d\Gamma_t,$$

where $\overline{\nu}$ is the x-component of the unit normal exterior ν.

Proof For the proof, see for example [5]. □

Lemma 5.2 *Let* $v \in H_0^1(\Omega) \cap H^2(\Omega)$. *Then for all* $i = 1, \dots, n$,

$$\frac{\partial v}{\partial y_i} = \eta_i \frac{\partial v}{\partial \nu}.$$

Proof For the proof, see for example [17]. □

Now we are in a position to show the uniform decay of the problem (1.2).

Theorem 5.3 *Take* $u_0 \in H_0^2(\Omega_0) \cap H^4(\Omega_0), u_1 \in H_0^1(\Omega_0)$ *and suppose that assumptions* (5.57) *and* (5.60) *hold. Then the strong solution of the system* (1.2) *satisfies*

$$E(t) \leq C E(0) e^{-\xi t}, \quad \forall t \geq 0,$$

where C *and* ξ *are positive constants.*

Proof Multiplying (1.2) by u_t, performing an integration by parts over Ω_t, and using Lemma 5.1, we obtain

$$\begin{aligned}
&\frac{d}{dt} E(t) + \|\nabla u_t(t)\|^2_{L^2(\Omega_t)} \\
&\quad = \frac{\gamma'}{2\gamma} M(\|\nabla u(t)\|^2) \int_{\Gamma_t} |\nabla u(t)|^2 (\overline{v} \cdot x) d\Gamma_t + \int_{\Gamma_t} \frac{\gamma'}{2\gamma} (\overline{v} \cdot x)(|u_t(t)|^2 + |\Delta u(t)|^2) d\Gamma_t,
\end{aligned}$$

where $E(t) = \frac{1}{2}(\|u_t(t)\|^2_{L^2(\Omega_t)} + \|\Delta u(t)\|^2_{L^2(\Omega_t)} + \|\nabla u_t(t)\|^2_{L^2(\Omega_t)})$.

Using (5.78), we obtain

$$\frac{\gamma'}{2\gamma} M(\|\nabla u(t)\|^2) \int_{\Gamma_t} |\nabla u(t)|^2 (\overline{v} \cdot x) d\Gamma_t + \int_{\Gamma_t} \frac{\gamma'}{2\gamma} (\overline{v} \cdot x)(|u_t(t)|^2 + |\Delta u(t)|^2) d\Gamma_t \leq 0.$$

By above two equations, we have the following result:

$$\frac{d}{dt} E(t) \leq -\|\nabla u_t(t)\|^2_{L^2(\Omega_t)}. \tag{5.80}$$

We consider the following functional:

$$\mathcal{L}(t) = E(t) + \varepsilon \Psi(t), \quad \text{where } \Psi(t) = \int_{\Omega_t} u u_t dx + \int_{\Omega_t} \nabla u_t \nabla u dx, \quad \forall \varepsilon > 0. \tag{5.81}$$

It is not difficult to see that $\mathcal{L}(t)$ verifies

$$k_0 E(t) \leq \mathcal{L}(t) \leq k_1 E(t), \quad \text{for } k_0, k_1 \text{ are positive constants.} \tag{5.82}$$

We will show later that the functional $\mathcal{L}$ satisfies the inequality of the following lemma.

Lemma 5.3 *Let* f *be a real positive function of class* $\mathbb{C}^1$. *If there exists positive constant* c_0 *such that* $f'(t) \leq -c_0 f(t)$, *then there exist positive constants* ξ *and* c_1 *such that* $f(t) \leq c_1 f(0) e^{-\xi t}$.

Proof The proof of this lemma is similar to the proof [13], we will omit. □

Taking the derivative of $\Psi(t)$ defined in (5.81) with respect to t, it follows that

$$\begin{aligned}\frac{d}{dt}\Psi(t) &= \|u_t(t)\|^2_{L^2(\Omega_t)} + \|\nabla u_t(t)\|^2_{L^2(\Omega_t)} - \|\Delta u(t)\|^2_{L^2(\Omega_t)}\\ &\quad - (\alpha + M(\|\nabla u(t)\|^2_{L^2(\Omega_t)}))\|\nabla u(t)\|^2_{L^2(\Omega_t)}\\ &\le -2E(t) + 2\|u_t(t)\|^2_{L^2(\Omega_t)} + 2\|\nabla u_t(t)\|^2_{L^2(\Omega_t)}\\ &\quad - \alpha\|\nabla u(t)\|^2_{L^2(\Omega_t)} - \widehat{M}(\|\nabla u(t)\|^2_{L^2(\Omega_t)})\\ &\le -2E(t) + 4E(0). \end{aligned} \tag{5.83}$$

Then, using (5.80) and (5.83), choosing ε small enough we get

$$\frac{d}{dt}\mathcal{L}(t) \le -2\varepsilon E(t) - (1-2\varepsilon)\|\nabla u_t(t)\|^2_{L^2(\Omega_t)} + 2\varepsilon\|u_t(t)\|^2_{L^2(\Omega_t)} \le -k_2\mathcal{L}(t), \tag{5.84}$$

where k_2 is a positive constant depend on ε, k_1.

Then, using Lemma 5.3, we obtain

$$\mathcal{L}(t) \le c_1\mathcal{L}(t)e^{-\xi t}, \quad \forall t \ge 0.$$

Therefore, using (5.84), we see that the following uniform decay result:

$$E(t) \le CE(0)e^{-\xi t}, \quad \forall t \ge 0.$$

The proof of Theorem 5.3 is completed. □

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