Asymptotic property for linear integro-differential systems

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Abstract


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1. Introduction and preliminaries

For the study of asymptotic property of linear integro-differential systems we recall two definitions: A differential system

\[ x' (t) = f (t, x), \quad x (t_0) = x_0, \quad f \in C (\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n) \]

is said to have asymptotic equilibrium if there exist a single \( \xi \in \mathbb{R}^n \) and \( r > 0 \) such that any solution \( x (t) \) with \( |x_0| < r \) satisfies

\[ x (t) = \xi + o(1) \quad \text{as} \quad t \to \infty \]

and for every \( \xi \in \mathbb{R}^n \), there exists a solution satisfying the above asymptotic relationship. Two differential systems

\[ x' (t) = f (t, x(t)) \quad \text{and} \quad y' (t) = g (t, y(t)) \]

are said to be asymptotically equivalent if, for every solution \( x(t) \), there exists a solution \( y(t) \) such that

\[ x(t) = y(t) + o(1) \quad \text{as} \quad t \to \infty \]

and conversely, for every solution \( y(t) \), there exists a solution \( x(t) \) satisfying the asymptotic relationship.

equivalence between two nonlinear differential systems by using a comparison principle. Moreover, Morchalo [8] and Nohel [9] established asymptotic equivalence between linear integro-differential systems and their perturbations by using the dominated convergence theorem and H"older inequality. Recently Choi et al. [4] obtained some asymptotic property in variation for nonlinear differential systems. For the asymptotic property for difference systems and Volterra difference systems, see [3,5,13].

We consider the linear integro-differential system

\[ x'(t) = A(t)x(t) + \int_0^t K(t,s)x(s)ds, \quad x(t_0) = x_0, \quad (1.1) \]

and its perturbation

\[ y'(t) = A(t)y(t) + \int_0^t K(t,s)y(s)ds + F(t), \quad y(t_0) = x_0, \quad (1.2) \]

where \( A(t) \) is an \( n \times n \) continuous matrix function on \( \mathbb{R}_+ = [0, \infty) \), \( K(t,s) \) is an \( n \times n \) continuous matrix function for \( 0 \leq s \leq t < \infty \), \( F \in C(\mathbb{R}_+, \mathbb{R}^n) \), and \( \mathbb{R}^n \) denotes the \( n \)-dimensional Euclidean space with any convenient norm \( | \cdot | \).

The unique solution \( y(t) \) of (1.2) is given by the variation of constants formula in [7, Theorem 1.7.1] as

\[ y(t) = R(t,t_0)x_0 + \int_{t_0}^t R(t,s)F(s)ds, \quad t \geq t_0, \quad (1.3) \]

where \( R(t,s) \) is the resolvent matrix which is the unique solution of the initial value problem

\[ \begin{align*}
\frac{\partial R(t,s)}{\partial s} + R(t,s)A(s) + \int_t^s R(t,\sigma)K(\sigma,s)d\sigma &= 0, \\
R(t,t) &= I, \quad \text{for } 0 \leq s \leq t < \infty.
\end{align*} \quad (1.4) \]

Also, the fundamental matrix solution \( \Phi(t,t_0,0) \) of (1.1) is given by

\[ \Phi(t,t_0,0) = \frac{\partial x(t,t_0,0)}{\partial x_0} \]

and \( \Phi(t,t_0,0) = R(t,t_0) \), where \( R(t,t_0) \) is the resolvent solution of (1.4) [7].

In Section 2, we investigate the asymptotic property of (1.1) and its perturbation (1.2) by means of the resolvent matrix \( R(t,s) \).

Note that the resolvent solution \( R(t,s) \) of (1.4) plays the same role as the fundamental matrix solution of \( x' = A(t)x \) when \( K(t,s) = 0 \), that is,

(i) \( R(t,s) \) is invertible.
(ii) \( R(t,t_0) = R(t,s)R(s,t_0) \) for all \( t \geq s \geq t_0 \).

However these properties do not hold in general when \( K(t,s) \neq 0 \). For instance,

(i) \( R(t,t_0) = 2e^{-4(t-t_0)} - e^{-3(t-t_0)} \) is not invertible at \( t = \ln 2 \), since \( R(\ln 2, 0) = 0 \).
(ii) \( R(t,t_0) = \frac{1}{2}(1+e^{-(t-t_0)}) \neq R(t,s)R(s,t_0) \) at \( t = 2 \ln 2, s = \ln 2 \), and \( t_0 = 0, \) since

\[ R(2\ln 2, 0) = \frac{1}{2} \left( 1 + \frac{1}{4} \right) = \frac{5}{8} \neq \frac{9}{16} = R(2\ln 2, 2\ln 2)R(2\ln 2, 0). \]

Thus, in Section 3, we use a useful equivalent system in [7, Theorem 1.7.2] to study the asymptotic property of (1.1) and (1.2):

Assume that there exists an \( n \times n \) continuous matrix function \( L(t,s) \) on \( \mathbb{R}_+^2 \) such that \( L_s(t,s) \) exists, is continuous and satisfies

\[ K(t,s) + \frac{\partial L}{\partial s}(t,s) + L(t,s)A(s) + \int_s^t L(t,u)K(u,s)du = 0. \quad (1.5) \]

Then (1.2) becomes the linear differential system

\[ x'(t) = A(t)x(t) + \int_0^t K(t,s)x(s)ds, \quad x(t_0) = x_0. \]
where $B(t) = A(t) - L(t, t)$ and $H(t) = F(t) + \int_{t_0}^{t} L(t, s) F(s) ds$ [7]. Moreover, the solution $u(t, t_0, x_0)$ of (1.6) satisfying $u(t_0, t_0, x_0) = x_0$ is given by

$$u(t) = Y(t)Y^{-1}(t_0)x_0 + \int_{t_0}^{t} Y(t)Y^{-1}(s)[L(s, t_0)x_0 + H(s)]ds,$$

where $Y(t)$ is a fundamental matrix solution of

$$y'(t) = B(t)y.$$

The following example illustrates the above facts.

**Example 1.1 ([7, Example 2.5.1]).** We consider the linear integro-differential equation

$$x' = -5x - 2 \int_{t_0}^{t} e^{-2(t-s)}x(s)ds, \quad x(t_0) = 0,$$

where $A(t) = -5$ and $K(t, s) = -2e^{-2(t-s)}$. Note that the solution of (1.9) is given by

$$x(t, t_0, x_0) = [2e^{-4(t-t_0)} - e^{-3(t-t_0)}]x_0, \quad t \geq t_0.$$

Then the linear differential equation equivalent to (1.9) with $F(t) = 0$ in (1.2) is given by

$$u' = B(t)u(t) + L(t, t_0)x_0 = -4u(t) - e^{-3(t-t_0)}x_0, \quad u(t_0) = x_0,$$

where $B(t) = A(t) - L(t, t) = -4$ and $L(t, s) = -e^{-3(t-s)}$ satisfies the following equation

$$K(t, s) + \frac{dL}{ds}(t, s) + L(t, s)A(s) + \int_{s}^{t} L(t, u)K(u, s)du$$

$$= -2e^{-2(t-s)} - \frac{e^{-3(t-s)}}{s} + 5e^{-3(t-s)} + 2 \int_{s}^{t} e^{-3(t-u)}e^{-2(u-s)}du$$

$$= 0.$$

Also we note that the solution $x(t)$ of (1.9) is given by

$$x(t, t_0, x_0) = R(t, t_0)x_0 = [2e^{-4(t-t_0)} - e^{-3(t-t_0)}]x_0, \quad t \geq t_0,$$

where $R(t, s) = 2e^{-4(t-s)} - e^{-3(t-s)}$ satisfies (1.4) with $A(s) = -5$ and $K(s, s) = -2e^{-2(s-s)}$ for each $t \geq s \geq t_0 \geq 0$.

2. Asymptotic property via resolvent matrices

In this section we investigate the asymptotic property of the linear integro-differential system (1.1) and its perturbation (1.2).

We need the following integral inequality and the variation of constants formula by means of the resolvent matrices to obtain our results.

**Lemma 2.1 ([10, Theorem 2.1]).** Let $u(t), a(t) \in C(\mathbb{R}_+, \mathbb{R}_+), b(t, s) \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ for $t_0 \leq s \leq t$ and $k \geq 0$ be a constant. If

$$u(t) \leq k + \int_{t_0}^{t} \left[ a(s)u(s) + \int_{t_0}^{s} b(s, \sigma)u(\sigma)d\sigma \right]ds,$$

for $t \in \mathbb{R}_+$, then

$$u(t) \leq k \exp \left( \int_{t_0}^{t} [a(s) + \int_{t_0}^{s} b(s, \sigma)d\sigma]ds \right),$$

for $t \in \mathbb{R}_+$.
Firstly, we give the necessary and sufficient condition for linear integro-differential system (1.1) to have asymptotic equilibrium via the resolvent solution of (1.4).

**Theorem 2.2.** (1.1) has asymptotic equilibrium if and only if \( \lim_{t \to \infty} R(t, t_0) \) exists and is invertible, where \( R(t, t_0) \) is the resolvent matrix solution of (1.4) for each \( t \geq t_0 \geq 0 \).

**Proof.** Suppose that (1.1) has asymptotic equilibrium. Then it is easy to show the existence of \( \lim_{t \to \infty} R(t, t_0) = R_\infty(t_0) \) for each \( t_0 \geq 0 \). Let \( E_i = (0, \ldots, 1, \ldots, 0)^T \) be the \( i \)th unit vector in \( \mathbb{R}^n \) for each \( i = 1, 2, \ldots, n \). Then there exist the solutions \( x(t, t_0, x_0) \) of (1.1) such that

\[
E_i = \lim_{t \to \infty} x(t, t_0, x_0) = \lim_{t \to \infty} R(t, t_0)x_0 = R_\infty(t_0)x_0, \quad i = 1, 2, \ldots, n.
\]

It follows that

\[
R_\infty(t_0)[x_{01} \cdots x_{0n}] = I,
\]
where \([x_{01} \cdots x_{0n}]\) is the inverse matrix of \( R_\infty(t_0) \). Thus \( R_\infty(t_0) \) is invertible.

Conversely, let \( \xi \) be any vector in \( \mathbb{R}^n \). Then there exists a solution \( x(t, t_0, x_0) \) of (1.1) with \( x_0 = R_\infty^{-1}(t_0)\xi \) such that the asymptotic property holds. This completes the proof. \( \square \)

**Corollary 2.3.** If (1.1) has asymptotic equilibrium, then there exists a positive constant \( M > 0 \) such that

\[
|R(t, s)| \leq M, \quad t \geq s \geq t_0 \geq 0.
\]

**Proof.** It follows from the existence of \( \lim_{t \to \infty} R(t, t_0) = R_\infty(t_0) \) for fixed \( t_0 \geq 0 \). \( \square \)

**Corollary 2.4.** Suppose that there exists a positive constant \( \alpha \) with \(| \det R(t, t_0) | \geq \alpha > 0 \) for each \( t \geq t_0 \) and \( \lim_{t \to \infty} R(t, t_0) = R_\infty \) exists. Then (1.1) has asymptotic equilibrium.

**Proof.** Note that

\[
\lim_{t \to \infty} | \det R(t, t_0) | = | \det \lim_{t \to \infty} R(t, t_0) |
\]

\[
= | \det R_\infty | \geq \alpha > 0.
\]

From the invertibility of \( R_\infty \) and Theorem 2.2, (1.1) has asymptotic equilibrium. \( \square \)

**Example 2.5.** To illustrate Theorem 2.2, we consider the linear integro-differential equation

\[
x'(t) = A(t)x(t) + \int_{t_0}^t K(t, s)x(s)ds = -\frac{1}{2}x(t) + \frac{1}{4} \int_{t_0}^t e^{-\frac{(t-s)}{2}} x(s)ds, x(t_0) = x_0, \quad t_0 \geq 0.
\]

(2.1)

Note that the solution of (2.1) is given by \( x(t, t_0, x_0) = \frac{\alpha}{2}[1 + e^{-(t-t_0)}] \). Thus we obtain that the resolvent solution \( R(t, t_0) \) of (1.4) is given by \( R(t, t_0) = \frac{1}{2}[1 + e^{-(t-t_0)}] \) and \( \lim_{t \to \infty} R(t, t_0) = \frac{1}{2} \neq 0 \). It follows from Theorem 2.2 that (2.1) has asymptotic equilibrium.

**Theorem 2.6.** Assume that both \( A(t) \) and \( \int_{t_0}^t K(t, s)ds \) belong to \( L^1(\mathbb{R}_+) \). Then (1.1) has asymptotic equilibrium.

**Proof.** (1.1) can be written as

\[
x(t, t_0, x_0) = x_0 + \int_{t_0}^t \left[ A(s)x(s) + \int_{t_0}^s K(s, \sigma)x(\sigma)d\sigma \right]ds.
\]

Since \( x(t, t_0, x_0) = R(t, t_0)x_0 \) for each \( x_0 \in \mathbb{R}^n \), \( R(t, t_0) \) satisfies the following integral equation

\[
R(t, t_0) = I + \int_{t_0}^t \left[ A(s)R(s, t_0) + \int_{t_0}^s K(s, \sigma)R(\sigma, t_0)d\sigma \right]ds.
\]

(2.2)
Lemma 2.1 and the assumption we obtain also has asymptotic equilibrium (1.1).

Theorem 2.2 (1.1) has asymptotic equilibrium and $F(t)$ has asymptotic equilibrium by Theorem 2.7.

Letting $|R(t, t_0)| = u(t)$, and

$$v(t) = 1 + \int_{t_0}^{t} \left[ |A(s)||R(s, t_0)| + \int_{t_0}^{s} |K(s, \sigma)||R(\sigma, t_0)|d\sigma \right] ds,$$

we have

$$v(t) = 1 + \int_{t_0}^{t} \left[ |A(s)|u(s) + \int_{t_0}^{s} |K(s, \sigma)||u(\sigma)|d\sigma \right] ds$$

$$\leq 1 + \int_{t_0}^{t} \left[ |A(s)||v(s)| + \int_{t_0}^{s} |K(s, \sigma)||v(\sigma)|d\sigma \right] ds.$$

From Lemma 2.1 and the assumption we obtain

$$v(t) \leq \exp \int_{t_0}^{t} \left[ |A(s)| + \int_{t_0}^{s} |K(s, \sigma)|d\sigma \right] ds$$

$$\leq \exp \int_{t_0}^{\infty} \left[ |A(s)| + \int_{t_0}^{s} |K(s, \sigma)|d\sigma \right] ds < \infty.$$

Then it is easy to see that $u(t) \leq v(t)$ for each $t \geq t_0$, and $v(t)$ is increasing and bounded. Furthermore, for any $t > t_1 > t_0$, we have

$$|R(t, t_0) - R(t_1, t_0)| \leq \int_{t_1}^{t} \left[ |A(s)||R(s, t_0)| + \int_{t_0}^{s} |K(s, \sigma)||R(\sigma, t_0)|d\sigma \right] ds$$

$$= v(t) - v(t_1).$$

This implies that, given any $\varepsilon > 0$, we can choose a $t_1 > 0$ sufficiently large so that

$$|R(t, t_0) - R(t_1, t_0)| < \varepsilon$$

for all $t > t_1$,

since $v(t)$ has the Cauchy property. Hence $R(t, t_0)$ converges to a constant $n \times n$ matrix $R_{\infty}(t_0)$ as $t \to \infty$.

Next there exists a constant $M > 0$ such that $|R(t, t_0)| \leq M$ for each $t \geq t_0$. Since $\int_{t_0}^{\infty} [|A(s)| + \int_{t_0}^{s} |K(s, \sigma)|d\sigma]ds < \infty$, we can choose $t_0 > 0$ large so that

$$\int_{t_0}^{\infty} \left[ |A(s)| + \int_{t_0}^{s} |K(s, \sigma)|d\sigma \right] ds < \frac{1}{M}.$$

Letting

$$P(t, t_0) = \int_{t_0}^{t} \left[ A(s)R(s, t_0) + \int_{t_0}^{s} K(s, \sigma)R(\sigma, t_0)d\sigma \right] ds,$$

we have

$$\lim_{t \to \infty} |P(t, t_0)| \leq \lim_{t \to \infty} \left[ \int_{t_0}^{t} \left[ |A(s)||R(s, t_0)| + \int_{t_0}^{s} |K(s, \sigma)||R(\sigma, t_0)|d\sigma \right] ds \right]$$

$$\leq M \lim_{t \to \infty} \left[ \int_{t_0}^{t} \left[ |A(s)| + \int_{t_0}^{s} |K(s, \sigma)|d\sigma \right] ds \right]$$

$$= M \int_{t_0}^{\infty} \left[ |A(s)| + \int_{t_0}^{s} |K(s, \sigma)|d\sigma \right] ds$$

$$< 1.$$

This implies that $\lim_{t \to \infty} R(t, t_0) = R_{\infty}$ is invertible. Then (1.1) has asymptotic equilibrium by Theorem 2.2.

The following theorem states that the property of having asymptotic equilibrium is preserved under the perturbation.

**Theorem 2.7.** Assume that (1.1) has asymptotic equilibrium and $F(t) \in L^1(\mathbb{R}_+)$. Then (1.2) also has asymptotic equilibrium.
Proof. Let \( y(t) \) be the solution of (1.2). Then the solution \( y(t) \) of (1.2) is given by
\[
y(t) = R(t, t_0)x_0 + \int_{t_0}^{t} R(t, s)F(s)ds, \quad t \geq t_0,
\]
from (1.4). Note that \( r(t) = \int_{t_0}^{t} R(t, s)F(s)ds \) has the Cauchy property by the boundedness of \( R(t,s) \) and \( \int_{t_0}^{\infty} |F(t)|dt < \infty \). It follows from Theorem 2.2 that the \( y(t) \) converges to a vector in \( \mathbb{R}^n \).

Conversely, let \( \xi \) be any vector in \( \mathbb{R}^n \). Since \( r(t) = \int_{t_0}^{t} R(t, s)F(s)ds \) has the Cauchy property, \( \lim_{t \to \infty} r(t) = r_\infty \) exists. Thus there exists a solution \( y(t) \) of (1.2) with the initial value \( y_0 = R_\infty^{-1}(\xi - r_\infty) \) such that the asymptotic relationship holds:
\[
y(t) = R(t, t_0)y_0 + \int_{t_0}^{t} R(t, s)F(s)ds
\]
\[
= R(t, t_0)R_\infty^{-1}(\xi - r_\infty) + r_\infty - \int_{t}^{\infty} R(\infty, s)F(s)ds
\]
\[
= \xi + o(1) \quad \text{as} \quad t \to \infty,
\]
since \( \int_{t}^{\infty} R(\infty, s)F(s)ds \to 0 \) as \( t \to \infty \). \( \square \)

Finally, we obtain asymptotic equivalence between (1.1) and (1.2).

**Theorem 2.8.** Assume that (1.1) has asymptotic equilibrium and \( F(t) \in L^1(\mathbb{R}_+) \). Then (1.1) and (1.2) are asymptotically equivalent.

**Proof.** Let \( x(t, t_0, x_0) \) be any solution of (1.1). Then there exists a solution \( y(t) \) of (1.2) with the initial value \( y_0 = x_0 - R_\infty^{-1}r_\infty \) such that
\[
x(t, t_0, x_0) = R(t, t_0)x_0
\]
\[
= y(t, t_0, x_0) - \int_{t_0}^{t} R(t, s)F(s)ds + R(t, t_0)R_\infty^{-1}r_\infty
\]
\[
= y(t, t_0, y_0) + o(1) \quad \text{as} \quad t \to \infty,
\]
where \( r_\infty = \int_{t_0}^{\infty} R(\infty, s)F(s)ds \).

Conversely, let \( y(t) \) be any solution of (1.2), there exists a solution \( x(t, t_0, x_0) \) of (1.1) with the initial \( x_0 = y_0 + R_\infty^{-1}r_\infty \) such that
\[
y(t, t_0, y_0) = R(t, t_0)y_0 + \int_{t_0}^{t} R(t, s)F(s)ds
\]
\[
= x(t, t_0, x_0) + \int_{t_0}^{t} R(t, s)F(s)ds - R(t, t_0)R_\infty^{-1}r_\infty
\]
\[
= x(t) + o(1) \quad \text{as} \quad t \to \infty. \quad \square
\]

We have the following example as an illustration of Theorem 2.8.

**Example 2.9.** We consider the two linear integro-differential equations
\[
x'(t) = -\frac{1}{2}x(t) + \frac{1}{4}\int_{t_0}^{t} e^{-\frac{(t-s)}{2}} x(s)ds, \quad x(t_0) = x_0, \quad (2.3)
\]
and
\[
y'(t) = -\frac{1}{2}y(t) + \frac{1}{4}\int_{t_0}^{t} e^{-\frac{(t-s)}{2}} y(s)ds + e^{\alpha t}, \quad y(t_0) = y_0, \quad (2.4)
\]
where \( A(t) = -\frac{1}{2} \), \( K(t, s) = \frac{1}{4}e^{-\frac{(t-s)}{2}} \) and \( F(t) = e^{\alpha t} \) with the negative constant \( \alpha \neq -1 \). Then (2.3) and (2.4) are asymptotically equivalent.
Proof. From an easy computation the solutions $x(t)$ and $y(t)$ of (2.3) and (2.4) are given by

$$x(t, t_0, x_0) = \frac{1}{2} [1 + e^{-(t-t_0)}] x_0$$

and

$$y(t, t_0, y_0) = \frac{1}{2} [1 + e^{-(t-t_0)}] y_0 + \frac{1}{2} [e^{at} - e^{a t_0}] + \frac{1}{2 (\alpha + 1)} [e^{at} - e^{(\alpha+1) t_0} e^{-t}],$$

respectively. Since (2.4) has asymptotic equilibrium, the asymptotic equivalence between (2.3) and (2.4) follows from Theorem 2.8 by putting $y_0 = x_0 + \frac{e^{a t_0}}{\alpha}$. □

3. Asymptotic property via equivalent system

We begin with the following theorem which shows asymptotic equilibrium for the linear integro-differential system (1.1) via the equivalent system (1.6) with $H(t) = 0$.

Theorem 3.1. Assume that $\lim_{t \to \infty} Y(t, t_0) = Y_\infty$ is an invertible constant matrix and $\int_{t_0}^{\infty} |Y^{-1}(s) L(s, t_0)| ds < 1$, where $Y(t, t_0) = Y(t) Y^{-1}(t_0)$. Then (1.2) with $F(t) = 0$ has asymptotic equilibrium.

Proof. Let $\xi$ be any vector in $\mathbb{R}^n$. Since

$$\int_{t_0}^{\infty} |Y^{-1}(s) L(s, t_0)| ds (\equiv Q)$$

exists. It follows from $|Q| < 1$ that $(I+Q)$ has an inverse matrix. Thus we can find the unique solution $x_0$ of the linear system $Y_\infty [I + Q] x_0 = \xi$ such that the solution of linear system is given by $x_0 = [I + Q]^{-1} Y_\infty^{-1} \xi$ via the invertible matrix of $Y_\infty [I + Q]$. Then there exists a solution $u(t) = u(t, t_0, x_0)$ of (1.6) with $F(t) = 0$ such that the following asymptotic relationship holds:

$$\lim_{t \to \infty} u(t) = \lim_{t \to \infty} \left[ Y(t) \left( I + \int_{t_0}^{t} Y^{-1}(s) L(s, t_0) ds \right) \right] x_0$$

$$= Y_\infty [I + Q] x_0$$

$$= Y_\infty [I + Q] [I + Q]^{-1} Y_\infty^{-1} \xi$$

$$= \xi.$$

Conversely, it easy to see that the solution $u(t)$ of (1.6) tends to a vector $\xi$ in $\mathbb{R}^n$ as $t \to \infty$. This completes the proof. □

The following example is an illustration of Theorem 3.1.

Example 3.2. Consider the ordinary linear differential system

$$u'(t) = B(t) u(t) + L(t, t_0) x_0$$

$$= \left( \begin{array}{cc} -e^{-t} & 0 \\ 2 + e^{-t} & 0 \end{array} \right) u(t) + \left( \begin{array}{cc} 0 \\ 3 \end{array} \right) x_0, \quad u(t_0) = x_0.$$

Then (3.1) has asymptotic equilibrium.

Proof. A fundamental matrix solution $Y(t, t_0)$ of the homogeneous differential system of (3.1) is given by

$$\left( \begin{array}{cc} 2 + e^{-t} & 0 \\ 2 + e^{-t} & 0 \end{array} \right).$$

Note that $\lim_{t \to \infty} Y(t, t_0) = Y_\infty(t_0)$ is invertible. Also, the solution $u(t, t_0, x_0)$ of (3.1) is given by

$$u(t) = Y(t, t_0) x_0 + \int_{t_0}^{t} Y(t, s) L(s, t_0) x_0 ds.$$
are given by

$$\begin{pmatrix} 2 + e^{-t} & 0 \\ 2 + e^{-t_0} & 1 \end{pmatrix} \left[ I + \begin{pmatrix} (1 + 2e^{t_0}) \ln\left(\frac{2 + e^{-t}}{2 + e^{-t_0}}\right) & 0 \\ 0 & \frac{1}{3}(e^{-3t_0} - e^{-3t}) \end{pmatrix} \right] x_0, \quad t \geq t_0.$$ 

Since $\int_0^\infty |Y^{-1}(s)L(s, 0)|ds = Q < 1$ and all conditions of Theorem 3.1 are satisfied, (3.1) has asymptotic equilibrium. □

**Corollary 3.3.** If two systems (1.1) and (1.2) have asymptotic equilibria, then (1.1) and (1.2) are asymptotically equivalent.

We need the following lemma to illustrate Corollary 3.3.

**Lemma 3.4** ([4]). (1.8) has asymptotic equilibrium if and only if $\lim_{t \to \infty} Y(t, t_0)$ exists and is invertible, where $Y(t, t_0)$ is a fundamental matrix solution of (1.8) for each $t \geq t_0 \geq 0$.

**Example 3.5.** We consider the linear integro-differential equation

$$x'(t) = A(t)x(t) + \int_{t_0}^t K(t, s)x(s)ds$$

and its perturbation

$$y'(t) = A(t)y(t) + \int_{t_0}^t K(t, s)y(s)ds + F(t)$$

where $A(t) = -\frac{1}{2}$, $K(t, s) = \frac{1}{4}e^{-\frac{1}{2}(t-s)}$ and $F(t) = e^{\alpha t}$ with the negative constant $\alpha \neq -1$. Then (3.2) and (3.3) are asymptotically equivalent.

**Proof.** Two linear differential equations equivalent to (3.2) and (3.3), respectively, are given by

$$v'(t) = B(t)v(t) + L(t, t_0)x_0$$

and

$$u'(t) = B(t)u(t) + L(t, t_0)u_0 + H(t)$$

where $B(t) = A(t) - L(t, t) = -\frac{3}{2}$, $L(t, s) = \frac{1}{4}(3 + e^{-\frac{1}{2}(t-s)})$ and $H(t) = F(t) + \int_{t_0}^t L(t, s)F(s)ds$. The solutions of (3.4) and (3.5) are given by

$$v(t, 0, v_0) = \frac{v_0}{2}(1 + e^{-t})$$

and

$$u(t, 0, u_0) = \frac{u_0}{2}(1 + e^{-t}) + \frac{1 + 2\alpha}{2\alpha(1 + \alpha)} e^{\alpha t} - \frac{e^{-t}}{2(1 + \alpha)} - \frac{1}{2\alpha}.$$
respectively. Thus both (3.4) and (3.5) have asymptotic equilibria by Lemma 3.4. In fact, for each solution \( v(t, 0, v_0) \) of (3.4), there exists a solution \( u(t) \) of (3.5) with the initial value \( u_0 = v_0 + \frac{1}{\alpha} \) such that the asymptotic relationship holds:

\[
u(t) = \frac{u_0}{2} (1 + e^{-t}) + \frac{1 + 2\alpha}{2\alpha(1 + \alpha)} e^{2t} - \frac{e^{-t}}{2(1 + \alpha)} - \frac{1}{2\alpha} \]

\[
v(t, 0, v_0) + \frac{1}{2\alpha} (1 + e^{-t}) + \frac{1 + 2\alpha}{2\alpha(1 + \alpha)} e^{2t} - \frac{e^{-t}}{2(1 + \alpha)} - \frac{1}{2\alpha} \]

\[
v(t, 0, v_0) + o(1) \quad \text{as} \quad t \to \infty.
\]

Also, the converse similarly holds. Hence two linear differential equations (3.4) and (3.5) are asymptotically equivalent. □

**Corollary 3.6.** In addition to the assumptions of Theorem 3.1 suppose that \( \int_{t_0}^\infty |H(s)|ds \) exists. Then (1.2) has asymptotic equilibrium.

**Proof.** Replace \( x_0 \) in the proof of Theorem 3.1 by

\[
x_0 = [I + Q]^{-1}[Y^{-1}\xi - v_\infty],
\]

where \( v_\infty = \int_{t_0}^\infty Y^{-1}(s)H(s)ds \). Then the rest of the proof is the same as in Theorem 3.1. □

**Corollary 3.7.** Assume that (i) \( \lim_{t \to \infty} Y(t, t_0) = Y_\infty \) is an invertible constant matrix and \( \int_{t_0}^\infty |Y^{-1}(s)L(s, t_0)|ds < 1 \) \( \int_{t_0}^\infty |H(s)|ds \) exists.

Then two systems (1.1) and (1.2) are asymptotically equivalent.

**Proof.** By assumption (i) and Theorem 3.1, (1.1) has asymptotic equilibrium. Also, from (ii) and Corollary 3.6, (1.2) has asymptotic equilibrium. It follows from Corollary 3.3 that (1.1) and (1.2) are asymptotically equivalent. □

**Example 3.8.** To illustrate Corollary 3.7, we consider the linear integro-differential equation

\[
x'(t) = A(t)x(t) + \int_{t_0}^t K(t, s)x(s)ds
\]

\[
= \left(\frac{1}{3} + \frac{e^{-t}}{2 + e^{-t}}\right)x(t) + \int_{t_0}^t K(t, s)x(s)ds, \quad x(0) = x_0,
\]

and its perturbation

\[
y'(t) = A(t)y(t) + \int_{t_0}^t K(t, s)y(s)ds + F(t)
\]

\[
= \left(\frac{1}{3} + \frac{e^{-t}}{2 + e^{-t}}\right)y(t) + \int_{t_0}^t K(t, s)y(s)ds + e^{\alpha t}, \quad y(0) = y_0 (= x_0),
\]

where \( F(t) = e^{\alpha t} \) with the negative number \( \alpha \neq -1 \) and \( K(t, s) \) is the continuous function for \( 0 \leq s \leq t \) satisfies (1.5) with \( A(t) = \left(\frac{1}{3} + \frac{e^{-\frac{t}{2}}}{2 + e^{-\frac{t}{2}}}\right) \) and \( L(t, s) = \frac{e^{-\left(\frac{t-s}{3}\right)}}{3} \) for each \( t \geq s \geq 0 \). Then (3.6) and (3.7) are asymptotically equivalent.

**Proof.** We consider two linear integro-differential equations

\[
u'(t) = B(t)v(t) + L(t, t_0)x_0
\]

\[
= \frac{-e^{-t}}{2 + e^{-t}}v(t) + \frac{e^{\left(\frac{-t}{3}\right)}x_0}{3}, \quad x(0) = x_0,
\]

and

\[
u'(t) = B(t)u(t) + L(t, t_0)x_0 + H(t)
\]
which are equivalent to

\[ F(t) = A(t) - L(t, t) = \frac{-e^{-t}}{2} \]

and

\[
H(t) = F(t) + \int_0^t L(t, s) F(s) ds \\
= e^{\alpha t} + \frac{1}{3(\alpha + 1)} [e^{\alpha t} - e^{-t}].
\]

It is easy to show that \( \lim_{t \to -\infty} Y(t, 0) = \lim_{t \to -\infty} (2 + e^{-t}) = \frac{2}{Q} = Y_{\infty} \) is invertible and \( \int_0^\infty |Y^{-1}(s)L(s, 0)| ds = \ln \frac{2}{Q} \). Thus it suffices to show that \( \int_0^\infty |H(s)| ds \) exists. In fact, it follows from the easy calculation that \( \int_0^\infty |H(s)| ds = \frac{2(3\alpha + 4)}{9(\alpha + 1)} \) exists. This implies that (3.6) and (3.7) are asymptotically equivalent by Corollary 3.6. This completes the proof. \( \square \)

To obtain a sufficient condition on asymptotic equivalence between (1.1) and (1.2) we need the system

\[ v'(t) = B(t)v(t) + L(t, t_0)x_0, \quad v(t_0) = x_0. \] (3.10)

**Theorem 3.9.** Assume that \( \lim_{t \to -\infty} Y(t, 0) = Y_{\infty} \) and \( \int_0^\infty Y^{-1}(s)L(s, 0) ds \) exist. Then (1.1) and (1.2) are asymptotically equivalent.

**Proof.** It suffices to prove that the two systems (3.10) and (1.6) which are equivalent to (1.1) and (1.2), respectively, are asymptotically equivalent. Let \( v(t, t_0, v_0) \) be any solution of (3.10) with the initial value \( v(t_0, t_0, v_0) = v_0 \). Then the solution \( u(t, t_0, u_0) \) of (1.6) is given by

\[
u(t, t_0, v_0) = Y(t, t_0)u_0 + \int_{t_0}^t Y(t, s)[L(s, t_0)x_0 + H(s)] ds \\
= v(t, t_0, v_0) + Y(t, t_0)(u_0 - v_0) + \int_{t_0}^t Y(t)Y^{-1}(s)H(s) ds.
\]

Thus there exists a solution \( u(t) \) of (1.2) with the initial value \( u_0 = v_0 - Y(t_0)p_{\infty} \) such that the asymptotic relationship holds:

\[
u(t) = v(t) + Y(t)[Y^{-1}(t)(u_0 - v_0) + p_{\infty}] \\
= v(t) + o(1) \quad \text{as} \quad t \to \infty,
\]

where \( p_{\infty} = \int_{t_0}^\infty Y^{-1}(s)H(s) ds \).

Conversely, let \( u(t) = u(t, t_0, u_0) \) be any solution of (1.6). Letting \( v_0 = u_0 + Y(t_0)p_{\infty} \), there exists a solution \( v(t, t_0, v_0) \) of (3.10) such that

\[
u(t, t_0, v_0) = v(t, t_0, v_0) + Y(t)[Y^{-1}(t_0)(-Y(t_0)p_{\infty}) + p(t)] \\
= v(t) + o(1) \quad \text{as} \quad t \to \infty,
\]

where \( p(t) = \int_{t_0}^t Y^{-1}(s)H(s) ds \). Hence (1.1) and (1.2) are asymptotically equivalent. This completes the proof. \( \square \)

**References**


